Discrete Analysis
DGCI 08

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INTRODUCTION

During the eighties Non-standard Analysis research team at *Advanced Mathematics Research Institute* (IRMA, Strasbourg) was interested in the global discretization of mathematics problem seen from Edward Nelson's point of view (Internal Set Theory, I.S.T.).

J. Harthong and G. Reeb realized that if this problem has some solution it must be solved using integers only: $\mathbb{R}$ is $\mathbb{Z}$ seen from far. Reeb showed the way solving the famous differential equation $y' = y$ using integers only.

This was the start of a huge activity called *Integer Calculus*, mainly around differential equations (around 1985-90), but which also led to an original approach of *Discrete Geometry*.

Introduction

It seemed that despite its simplicity Euclidean Geometry should not be neglected in this quest of Discrete Theories. Moreover Discrete Geometry seemed to be a *hot* subject to which applied mathematicians and computer scientists deserved much attention.

After 1990 the project of building a Discrete Analysis Theory was abandoned by NSA team and research on Discrete Geometry moved to a computer science team: LSIIT also in Strasbourg.

Its maybe now the time to have a look at this old piece of work, made mostly with A. Troesch, which was a first step toward *Discrete Analysis*.

I thank the organizers for the opportunity to present this subject.

Introduction

Discrete curves analysis

In order to be useful and efficient (for *computer science*) this analyzis should have the following properties

- remain close to integers,
- able to handle finer and finer *structures*

- First property allows inspection and use of arithmetics
- Second property permits to go to smaller and smaller details like with usual Mathematical Analysis.

Non-standard Analysis specificity

Infinitesimals or orders of magnitude rigorously defined in Non-standard Analysis can be mimicked by *sequences of subgroups* of $\mathbb{Z}$ (or $\mathbb{Z}^n$) to obtain an analog of Harthong's thesis $\mathbb{R}$ is $\mathbb{Z}$ seen from far.

The aim

We want to be able to work with *tangent discrete lines*, *tangent discrete parabolas*, or more generally *tangent discrete polynomials* of discrete curves.

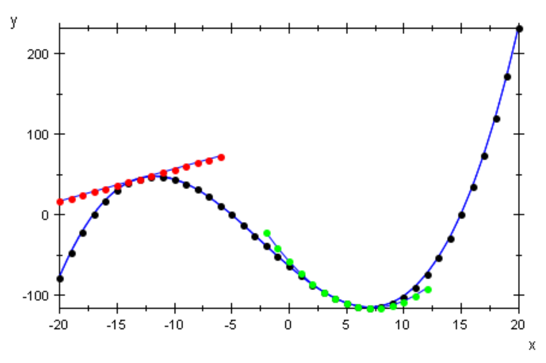

Figure: Discrete analysis of discrete curve

DIGITS CALCULUS

Digits calculus

Integer part function

Use of *Integer part* (or *floor*) function produces discretized functions and gives bound for error values, but this bound is to *coarse* and does not see *number theoretic properties*.

Desired properties of errors

Beside recovering right values, these quantities should be *rational* and inform us about :

- periodicities
- inner arithmetics (prime factors, quadratic residues,...)

of a problem.

Discrete multi-scales

It is convenient, in order to be able to discretize a function with respect to several scales, to introduce subgroups of $\mathbb{Z}$.
If $\omega > 1$ is integer, the chain :

$$\mathbb{Z}\omega^k \subset \mathbb{Z}\omega^{k-1} \subset \cdots \subset \mathbb{Z}$$

leads to the usual scale of magnitudes $1, \omega, \ldots \omega^k \ldots$

The same can be done in higher dimensions.

Digits Calculus

Two views of a same concept

Using former chains of subgroups any integer x may be denoted

$$x = c_k\omega^k + c_{k-1}\omega^{k-1} + \cdots + c_1\omega + c_0$$

for convenient k where, moreover, conditions

$$\forall i \in [0, k], \quad 0 \leq c_i < \omega$$

can be imposed.

This is the usual positional number system where c_i are the digits of $x$with respect to radix ω.

ω-rationals

Dividing integers $x = c_k\omega^k + c_{k-1}\omega^{k-1} + \cdots + c_1\omega + c_0$ by ω^k leads to rationals

$$x = c_0 + \frac{c_1}{\omega} + \cdots + \frac{c_{k-1}}{\omega^{k-1}} + \frac{c_k}{\omega^k}$$

which are closer our way of computing but are equivalent to the all integer former view.

Digits Calculus

Using multi-scales

If $e(x, y, z, \dots)$ is an expression, Digits Calculus is the study of its formal dependence on the digits of $x, y, z, \dots$.

Remark

When Digits Calculus started in Strasbourg, interest of multi-scale discretization was not clearly recognized; Discrete Geometry developed meanwhile has been influencial.

Digits Calculus of $x \mapsto x^2$

x^2 expressed with x digits

If two digits of x are used, that is $x = c_0 + \frac{c_1}{\omega}$, we have

$$x^2 = c_0^2 + 2\frac{c_0\,c_1}{\omega} + \frac{c_1^2}{\omega^2}$$

Dependencies on digits

Several dependencies $c_0 + \frac{c_1}{\omega} \to c_0^2, \quad c_0 + \frac{c_1}{\omega} \to c_0^2 + 2\frac{c_0\,c_1}{\omega}, \ldots$ can be studied.

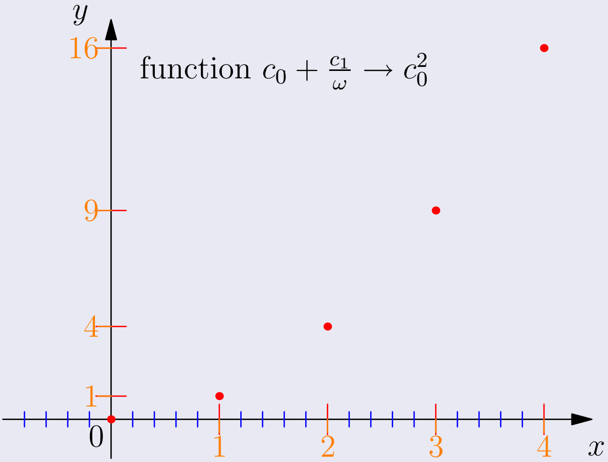

Figure: Coarse view of x^2

On the following picture next dependence is shown.

Digits Calculus of $x \mapsto x^2$

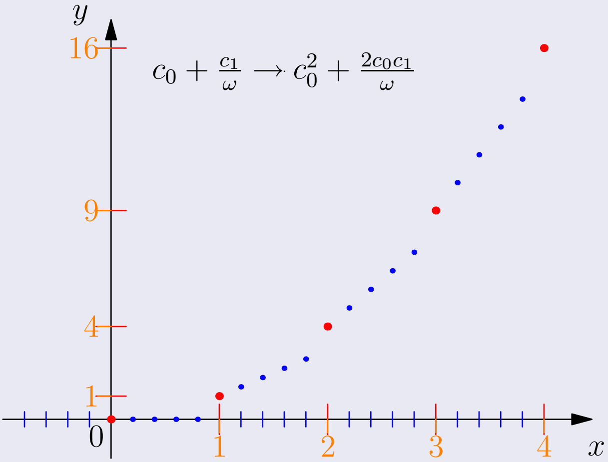

Figure: Finer view of x^2.

Of course next dependence gives x^2 on the finer scale $\dfrac{k}{\omega}$, $k \in \mathbb{Z}$.

Digits Calculus of $x \mapsto x^2$

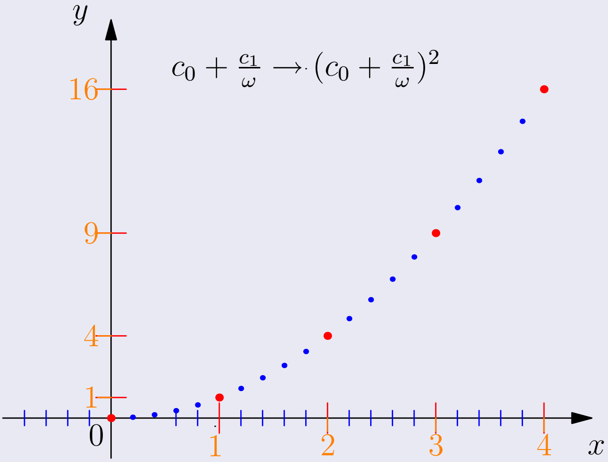

Figure: View of x^2 on finer grid.

Values on finer grid given by second and third dependence are *rational number*, let us discretize them taking *integer part* afterwards denoted by brackets [ ].

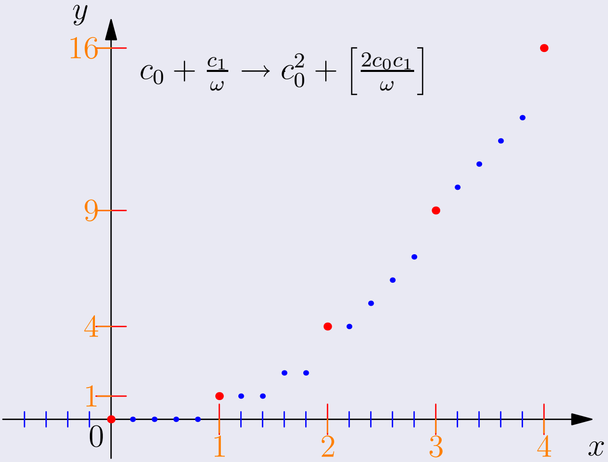

Figure: Coarse discretization of x^2.

Digits Calculus of $x \mapsto x^2$

Strange discretization

We can also consider function $c_0 + \frac{c_1}{\omega} \to c_0^2 + \left[\frac{2c_0c_1 + \left[\frac{c_1^2}{\omega}\right]}{\omega}\right]$

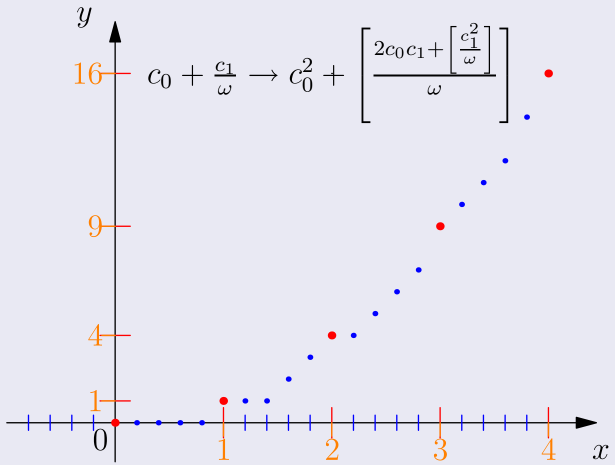

Figure: Strange discretization of x^2.

Digits Calculus of $x \mapsto x^2$

Let us have a look at usual discretization of x^2.

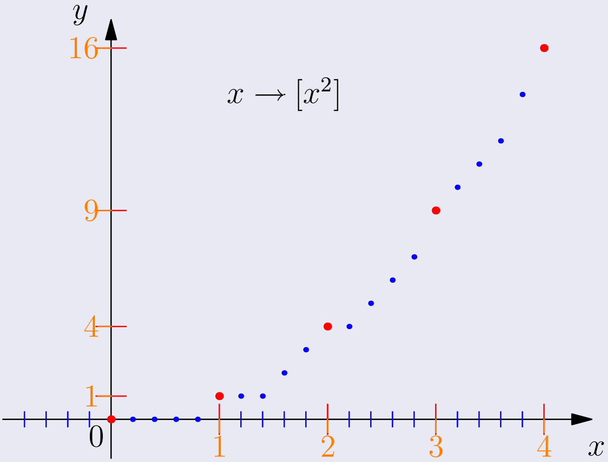

Figure: Usual discretization of x^2.

Strange and usual discretizations are identical

We see, on this example, the identity :

$$\left[(c_0 + \frac{c_1}{\omega})^2\right] = c_0^2 + \left[\frac{2c_0c_1 + \left[\frac{c_1^2}{\omega}\right]}{\omega}\right] \qquad (1)$$

We shall see that the complicated right side of this identity can be generalized and describes inner machinery of discretization.

Expressions appearing in equation (1) are Ehrhart quasi-polynomials, which will be encountered again in the following.

Carry report

As seen in previous example, c_0, c_1 are digits for radix ω, but c_0^2, $2c_0c_1$, c_1^2 are generally not digits; carry reports have to be done in order to obtain digits of expression $c_0^2 + \frac{2c_0c_1}{\omega} + \frac{c_1^2}{\omega^2}$. There are two carries whose algebraic expression is :

$$\left(c_0 + \frac{c_1}{\omega}\right)^2 = c_0^2 + \left[\frac{2c_0c_1 + \left[\frac{c_1^2}{\omega}\right]}{\omega}\right] + \frac{1}{\omega}\left\{\frac{2c_0c_1 + [\frac{c_1^2}{\omega}]}{\omega}\right\} + \frac{1}{\omega^2}\left\{\frac{c_1^2}{\omega}\right\}$$

where *curly brackets* $\left\{\frac{u}{\omega}\right\}$ denotes $u \mod \omega$.

Digits Calculus

Last two digits of radix ω writing of $\left(c_0 + \frac{c_1}{\omega}\right)^2$ are :

$$\left\{\frac{2c_0c_1 + [\frac{c_1^2}{\omega}]}{\omega}\right\} \text{ and } \left\{\frac{c_1^2}{\omega}\right\}$$

and we have the bound :

$$\frac{1}{\omega}\left\{\frac{2c_0c_1 + [\frac{c_1^2}{\omega}]}{\omega}\right\} + \frac{1}{\omega^2}\left\{\frac{c_1^2}{\omega}\right\} \leq \frac{\omega^2 - 1}{\omega^2} < 1$$

This bound shows that integer part of $\left(c_0 + \frac{c_1}{\omega}\right)^2$ is $c_0^2 + \left[\frac{2c_0c_1 + \left[\frac{c_1^2}{\omega}\right]}{\omega}\right]$ proving equation (1).

Remark

This integer part is generally *not* the *first* digit of $\left(c_0 + \frac{c_1}{\omega}\right)^2$, but it can be expressed as $e_k\omega^k + \cdots + e_1\omega + e_0$ with $0 \leq e_i < \omega$, for a convenient value of k.
Main point is its being *integer*.

Periodicity

Both functions

$$\left\{\frac{2c_0c_1 + [\frac{c_1^2}{\omega}]}{\omega}\right\} \text{ and } \left\{\frac{c_1^2}{\omega}\right\}$$

are periodical of period ω.

POLYNOMIAL CARRY PROPAGATION

Polynomial carry propagation

Rational polynomials

We limit the generalization to polynomials of $\mathbb{Q}[x]$ described as:

$$p(x) = \frac{1}{\omega}(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)$$

where $a_i \in \mathbb{Z}$ and $\omega > 1$ is also integer taken as radix.

Polynomial carry propagation

x digits

We suppose development of x with respect to radix ω can be written as:

$$x = c_0 + \frac{c_1}{\omega} + \cdots + \frac{c_r}{\omega^r}$$

Polynomial carry propagation

Entire development of $p(x)$ is complicated, it is better to operate recursively digit by digit letting:

$$x = c_0 + \frac{h}{\omega}, \quad \text{where} \quad h = c_1 + \frac{c_2}{\omega} + \cdots + \frac{c_r}{\omega^{r-1}}$$

that is we study first carry propagation associated with c_0.

Polynomial carry propagation

A useful identity

Identity:

$$\sum_{k=0}^{n} a_k (c_0 + \frac{h}{\omega})^k = \sum_{p=0}^{n} \left(\frac{h}{\omega}\right)^p \sum_{k=p}^{n} \binom{k}{p} a_k c_0^{k-p} \qquad (2)$$

gives this propagation. Equation (2) can be applied again with $h = c_1 + \dfrac{h}{\omega}$ to get second propagation, and so on.

Cubical carry propagation example

If $p(x) = \frac{1}{\omega}(a_0 + a_1 x + a_2 x^2 + a_3 x^3)$ and $x = c_0 + \frac{c_1}{\omega}$, composition evaluates to:

$$p(x) = \frac{1}{\omega}\left(p(c_0) + \frac{p'(c_0)}{1!}\frac{c_1}{\omega} + \frac{p''(c_0)}{2!}\frac{c_1^2}{\omega^2} + \frac{p'''(c_0)}{3!}\frac{c_1^3}{\omega^3}\right) \quad (3)$$

Discrete tangent

First order carry report gives expression of *discrete tangent* at point c_0:

$$\left\lfloor \frac{p(c_0) + \left\lfloor \frac{p'(c_0)c_1}{\omega} \right\rfloor}{\omega} \right\rfloor \quad (4)$$

Cubical carry propagation example

Discrete second order approximation

It is given by the formula:

$$\left\lfloor \frac{p(c_0) + \left\lfloor \frac{p'(c_0)c_1 + \left\lfloor \frac{p''(c_0)c_1^2}{2\omega} \right\rfloor}{\omega} \right\rfloor}{\omega} \right\rfloor \qquad (5)$$

DISCRETE TAYLOR FORMULA

Discrete Taylor Formula

Rational polynomial with two levels

Former equation (3) can be easily generalized to degree n polynomials giving:

$$p(x) = \frac{1}{\omega}(p(c_0) + \frac{p'(c_0)}{1!}\frac{c_1}{\omega} + \cdots + \frac{p^{(n)}(c_0)}{n!}\frac{c_1^n}{\omega^n}) \qquad (6)$$

Simplified notations

Notations for *carry report* can be simplified introducing the sequence $\{e_i\}$ of *errors* and the sequence of *carries* $[r_i]$ for $i = n, n-1, \ldots, 1, 0$.

Discrete Taylor Formula

Carries and errors

Let

$$e_n = \{\frac{p^{(n)}(c_0)}{n!}c_1^n/\omega\}$$

and

$$r_n = [\frac{p^{(n)}(c_0)}{n!}c_1^n/\omega],$$

where, I recall, $\{u/\omega\}$ means $u \bmod \omega$ and $[u/\omega]$ means $u \operatorname{div} \omega$ (the *floor* function.)

Discrete Taylor Formula

Carries and errors

Then, for $n > i \geq 0$, we let:

$$e_i = \{(\frac{p^{(i)}(c_0)}{i!}c_1^i + r_{i+1})/\omega\}$$

and

$$r_i = [(\frac{p^{(i)}(c_0)}{i!}c_1^i + r_{i+1})/\omega],$$

Discrete Taylor Formula

Applied on equation (6) this process gives its w-rational development (two scales):

$$p(x) = \mathcal{P}(c_0, c_1) + \frac{1}{\omega}(e_0 + \frac{e_1}{\omega} + \cdots + \frac{e_n}{\omega^n})$$

Where:

$$\mathcal{P}(c_0, c_1) = \left[\frac{p(c_0) + \left[\frac{p'(c_0)c_1 + \left[\cdots + \left[\frac{\frac{p^{(n)}(c_0)}{n!}c_1^n}{\omega}\right]\right]}{\omega}\right]}{\omega}\right]$$

And:

$$0 \leq e_i < \omega \quad \text{for} \quad i = 0, 1, \ldots, n.$$

Polynomial carry propagation

As in the case of function x^2, sum $\frac{1}{\omega}(e_0 + \frac{e_1}{\omega} + \cdots + \frac{e_n}{\omega^n})$ is strictly bounded by 1, showing that $\mathcal{P}(c_0, c_1) = [p(x)]$, the *integer part* of rational polynomial $p(x)$.

Carry propagation process collects *integer parts* along the w-rational development to gather them on the leading digit.

Polynomial carry propagation numerical example

$$
\begin{aligned}
\frac{1}{5}\left(2+\frac{4}{5}\right)^2 &= \frac{1}{5}\left(4+\frac{16}{5}+\frac{16}{25}\right) \\
&= \frac{1}{5}\left(4+\frac{16}{5}+\frac{3\times 5+1}{25}\right) \\
&= \frac{1}{5}\left(4+\frac{16}{5}+\frac{3}{5}+\frac{1}{25}\right) \\
&= \frac{1}{5}\left(4+\frac{19}{5}+\frac{1}{25}\right) \\
&= \frac{1}{5}\left(4+\frac{3\times 5+4}{5}+\frac{1}{25}\right) \\
&= \frac{1}{5}\left(4+3+\frac{4}{5}+\frac{1}{25}\right) \\
&= \frac{1}{5}\left(7+\frac{4}{5}+\frac{1}{25}\right) \\
&= 1+\frac{2}{5}+\frac{4}{25}+\frac{1}{125}
\end{aligned}
$$

Discrete Taylor Formula

Increasing formulas $\left[\frac{p(c_0)}{\omega}\right]$, $\left[\frac{p(c_0) + \left[\frac{p'(c_0)c_1}{\omega}\right]}{\omega}\right]$, etc. give local discrete approximations of order $0, 1, 2, \ldots$ of discrete polynomial $[p(x)]$

This is the discrete analog of Taylor formula for *two* scales.

Discrete Taylor Formula

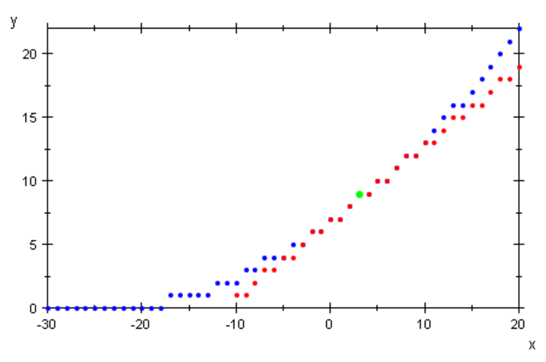

Figure: Discrete analysis of discrete curve

Discrete Taylor Formula

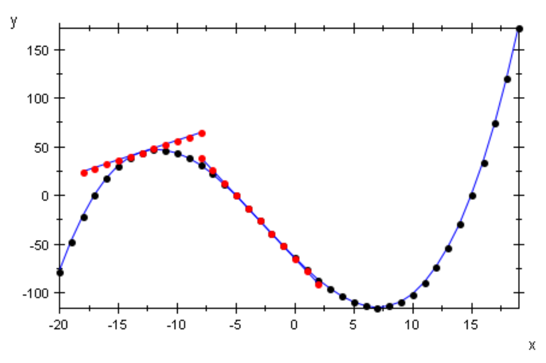

Figure: Discrete analysis of discrete curve

EHRHART QUASI-POLYNOMIALS

Ehrhart quasi-polynomials

E. Ehrhart

Eugène Ehrhart (1906-2000) was a Strasbourg mathematician who discovered the relationship between the volume of an integral polyhedron and the number of integer points it contains, the relation being a kind of polynomial.

Combinatorial definition of quasi-polynomial

If $p_0, p_1, \ldots, p_{\omega-1}$ is a set of ω polynomials of degree d, function q defined by:

$$\forall n \in \mathbb{Z} \quad q(n) = p_{n \bmod \omega}(n)$$

is called a quasi-polynomial of *constituents* p_i.

Quasi-polynomials

Ehrhart polynomial

The *Ehrhart polynomial* is the special quasi-polynomial related to polyhedra; it is the object of a very active research.

Simple examples of quasi-polynomials

Discrete lines are quasi-polynomials as is the case also for $[p(x)],\ p \in \mathbb{Q}$. But these q-polynomials are rather simple: their constituents are the same within *translation*; they only differ by their constant coefficient.

Properties

Quasi-polynomials have nice properties, particularly they possess *generating functions.*

Further reading

Beside Ehrhart's book: *Polynômes arithmétiques et méthode des polyèdres en combinatoire*, one can consult R. Stanley's one: *Enumerative Combinatorics.* A.I. Barvinok and Ph. Clauss are good specialists interested in the computing of the Ehrhart polynomial and its applications.

PARABOLIC DIOPHANTINE EQUATION

Parabolic Diophantine equation

Motivation

Arithmetics of parabolas is known but it is rather hard finding short elementary treatment of this subject.

Rational parabola

If a, b, c, ω are four integers, $\omega > 1$ being prime, the rational parabola is defined by $y = \frac{1}{\omega}(ax^2 + bx + c)$; this is a special case of the previous rational polynomials.

Parabolic Diophantine equation

Classical result

It is well known that Diophantine equation

$$ax^2 + bx + c = \omega y \tag{7}$$

has solutions iff equation

$$ax^2 + bx + c \equiv 0(\omega) \tag{8}$$

has at least one solution in the cyclic group $\mathbb{Z}/\omega\mathbb{Z}$.

Parabolic Diophantine equation

If $\delta = b^2 - 4ac$ is a quadratic residue mod ω then equation (7) has at least one solution; thus there exist integers α and ℓ such that:

$$a\alpha^2 + b\alpha + c = \ell\omega$$

Let α be such a solution of equation (7), then all solutions of equation (6) are parametrized in the following way:

$$\forall k \in \mathbb{Z} \quad (\alpha + \omega k, \ell + (2a\alpha + b)k + a\omega k^2).$$

Parabolic Diophantine equation

Example

Let us consider Diophantine equation $3x^2 - 5x + 10 = 13y$;
$\delta = -95 \equiv 9 \equiv 3^2 \mod 13$, thus δ is a quadratic residue mod $\omega = 13$. We find $\alpha = 9$ ($\alpha = 10$ is second solution), and $\ell = 16$; thus all solutions of previous equation are

$$(9 + 13k, 16 + 49k + 39k^2), \quad k \in \mathbb{Z}$$

and

$$(10 + 13k, 20 + 55k + 39k^2), \quad k \in \mathbb{Z}.$$

Four solutions of this equation, for $k = -1, k = 0$, appear on the following picture.

Parabolic Diophantine equation

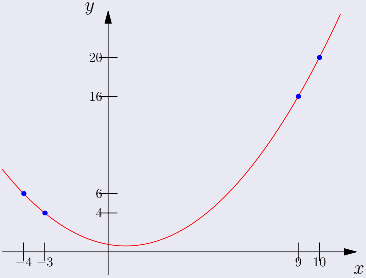

Figure: Diophantine equation $3x^2 - 5x + 10 = 13y$.

Parabolic Diophantine equation

Remark

Structure of Diophantine equation $ax^2 + bx + c = \omega y$ depends highly on arithmetics of coefficients a, b, c, ω.

Parabolic Diophantine equation

Definition

Discrete parabola can be defined as the graph of function $y = \left[\dfrac{ax^2 + bx + c}{\omega} \right]$ or as the set of integer points satisfying inequalities:

$$\gamma \leq ax^2 + bx - \omega y < \gamma + \epsilon$$

with $\epsilon = \omega$.

Parabolic Diophantine equation

Both definitions show that the periodic function $err(x) = \left\{ \dfrac{ax^2 + bx + c}{\omega} \right\}$, seen as *error*, or the *quadratic modular form* $ax^2 + bx - \omega y$, separates the structure in a finite number of *parabolic lines* containing solutions of a precise Diophantine equation $ax^2 + bx - \omega y = e$ where e is one of the values taken by previous *error* function. These values index these parabolic lines.

Parabolic Diophantine equation

Periodicity of function $err(x)$ shows that if $x_1 = x_0 + \omega$ and if (x_0, y_0) belongs to parabolic line $ax^2 + bx - \omega y = e$, then $(x_1, y_0 + 2ax_0 + b + \omega)$ is located on the same polynomial.

This shows that discrete parabola is a *quasi-polynomial*.

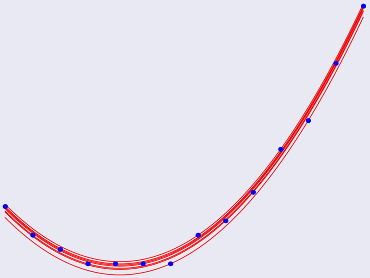

Figure: Diophantine view of discrete parabola

RECOGNITION OF DISCRETE PARABOLA

Recognition of discrete parabola

Let $p(x) = [\frac{1}{\omega}(ax^2 + bx + c)]$, where a, b, c, ω are integers, a short piece of a discrete parabola; we want to *recognize the parameters* from the knowledge of this limited discrete curve.

A useful result

If q(x) is a *quasi-polynomial* of degree 2, whose constituents have the same coefficients a of degree 2 and b of degree 1, then *divided differences* of step $h > 1$ belong to a discrete line of slope $2a$.

Recognition of discrete parabola

Recognition of this discrete line gives value of a and ω.

Subtraction of $[\frac{ax^2}{\omega}]$ from $p(x)$ gives another discrete line whose recognition gives parameter b.

Let us apply this to example appearing on the following picture of an arc of discrete parabola (about 100 pixels long).

Recognition of discrete parabola

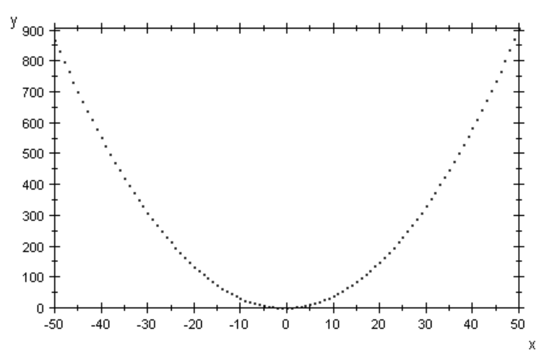

Figure: Arc of discrete parabola to recognize

Recognition of discrete parabola

Using a step of $h = 14$ the set of divided differences is represented on next image.

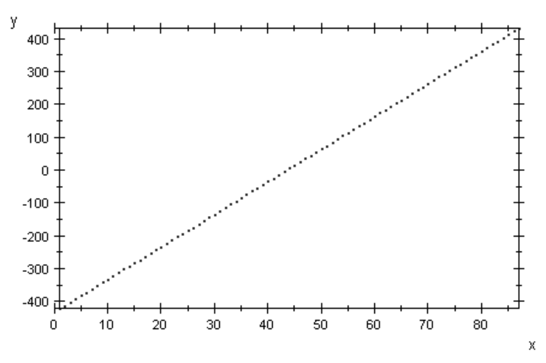

Figure: Divided differences applied to the former arc of discrete parabola

To recognize this discrete line convex hull is used. Slope of longest sides gives:

$$\frac{a}{\omega} = \frac{11}{31},$$

and that's it!

CONCLUSION

Conclusion

I always remember with great pleasure time spent at Strasbourg during the eighties at IRMA and the first half of nineties at LSIIT.

Work at IRMA with very qualified colleagues allowed the birth of many subjects, some not completely Non-standard.

LSIIT activities led to important results in Discrete Geometry, making further investigations like those of this talk, much easier.

I don't forget CPE, which was the school and workplace of Oscar Figueiredo, one of my pupils, who left us so tragically.