

Geometric representations and algorithms

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Geometric representations in testing/proving graph properties

Rubber bands: planarity, connectivity, random walk, maximum cut

Touching circles: planar separator, random walk

Orthogonal: stable set, clique, chromatic number, connectivity, treewidth

Small stretch: multicommodity flows, bandwidth

Nullspace: series-parallel, planarity, linkless embedding

Projective distance: outerplanarity, planarity, bisection

Independence preserving: α -critical graphs

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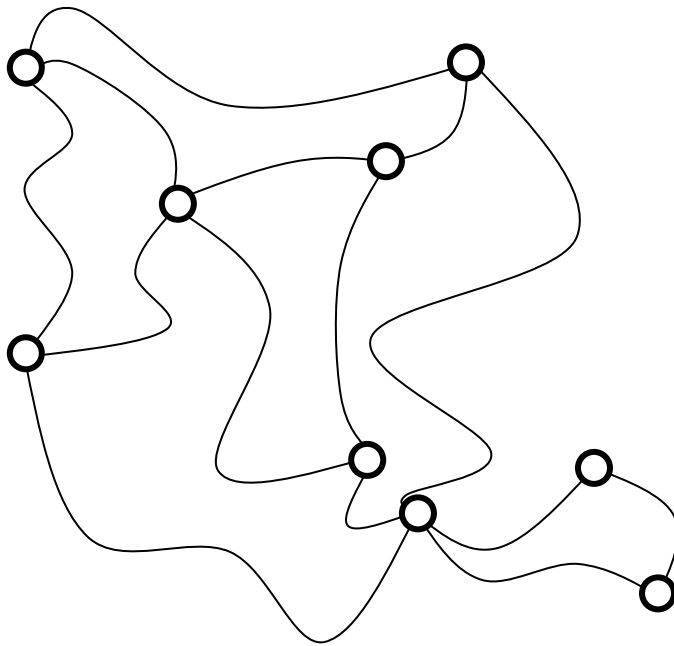
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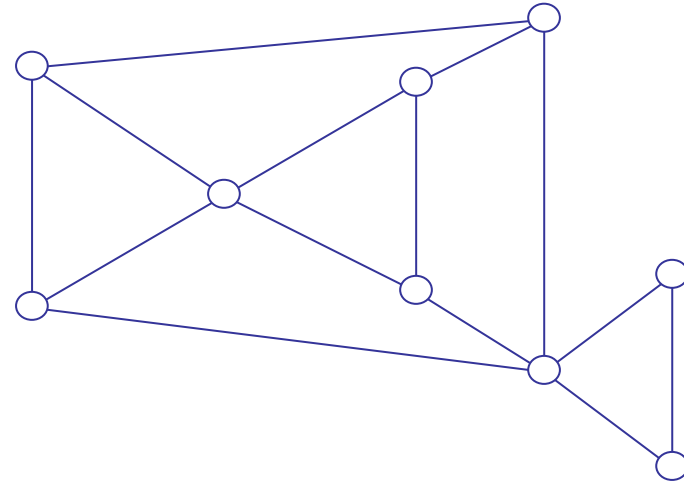
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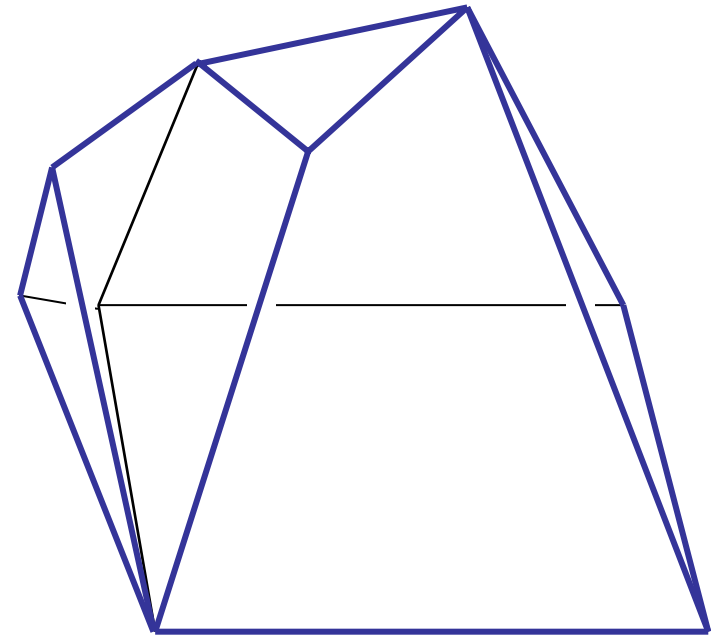
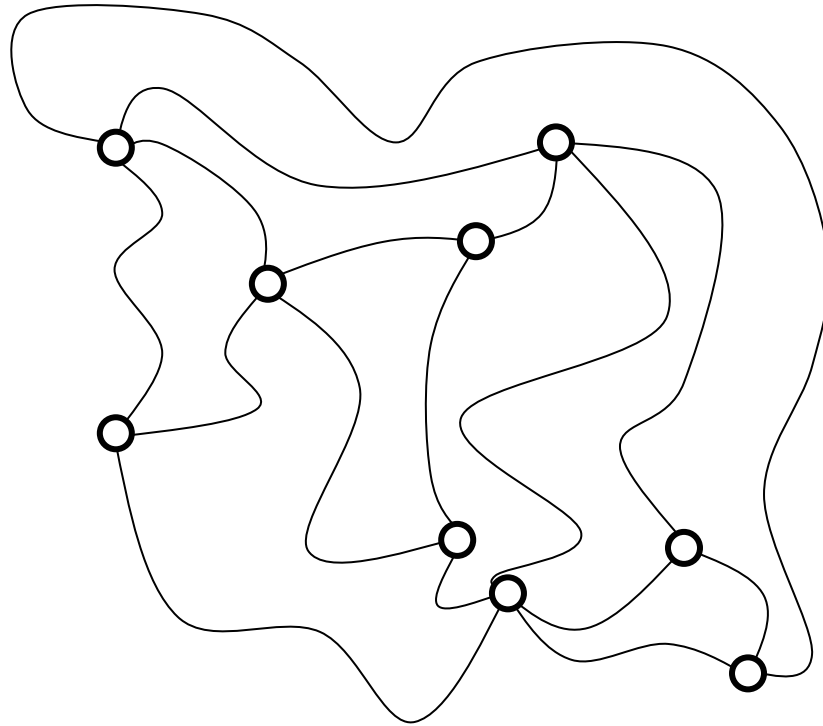


planar graph



Every planar graph can be drawn
in the plane with straight edges

Fáry-Wagner



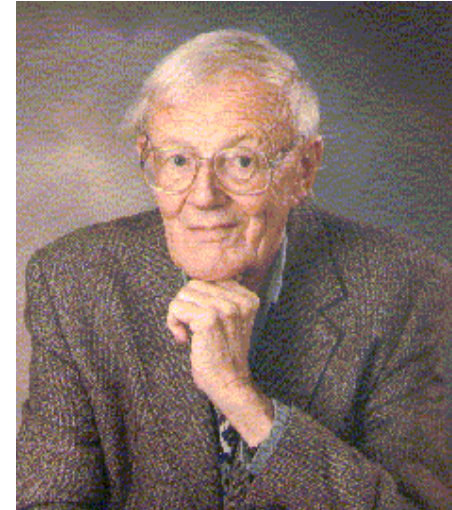
3-connected planar graph

Every 3-connected planar graph
is the skeleton of a convex 3-polytope.

Steinitz 1922

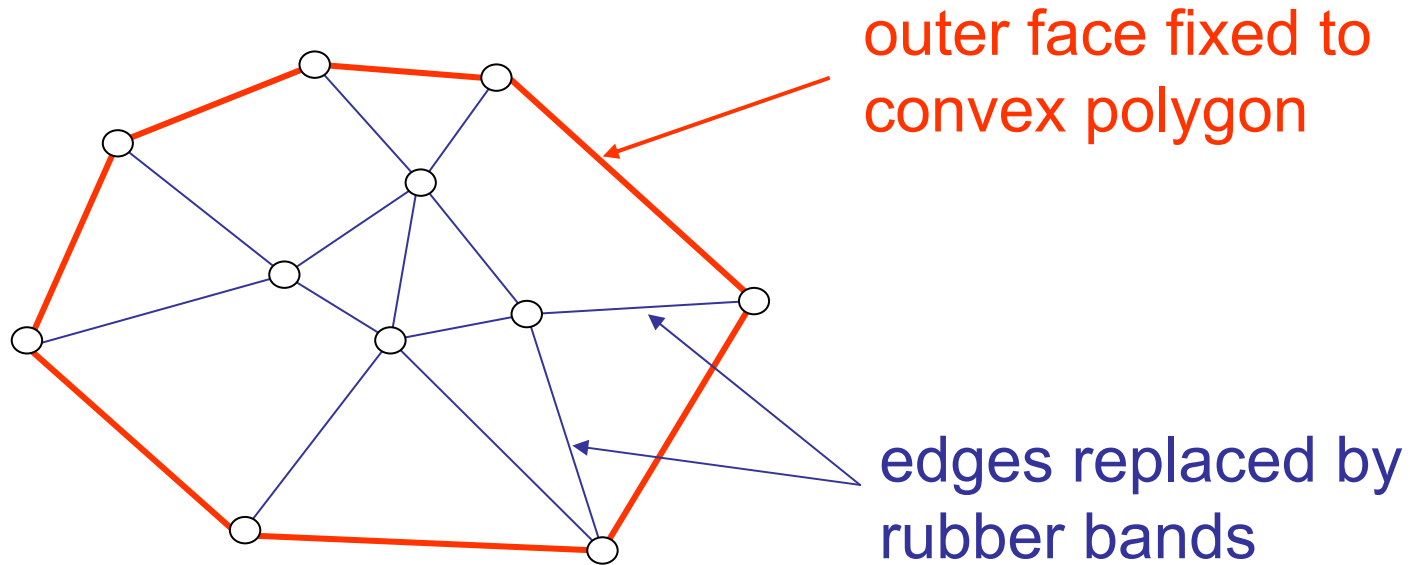
Rubber bands and planarity

Tutte (1963)



Every 3-connected planar graph can be drawn with straight edges and convex faces.

Rubber bands and planarity

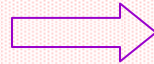


Energy:
$$E = \sum_{ij \in E} (u_i - u_j)^2$$

Equilibrium:
$$u_i = \frac{1}{d_i} \sum_{j \in N(i)} u_j$$

[Demo](#)

G 3-connected planar



rubber band embedding is planar

Tutte

(Easily) polynomial time
computable

Lifts to Steinitz
representation if
outer face is a triangle

Maxwell-Cremona

Connectivity of graphs

G : graph A, B : sets of nodes, $|A|=|B|=k$

A, B k -connected in G :  k disjoint paths between A and B

A, B are k -connected in G iff they cannot be separated by $k-1$ nodes.

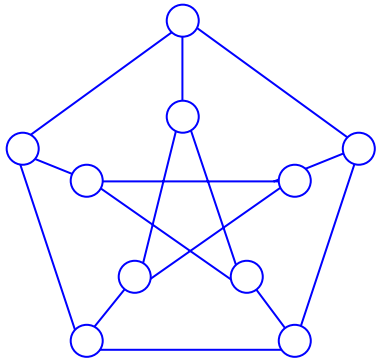
Menger

1927

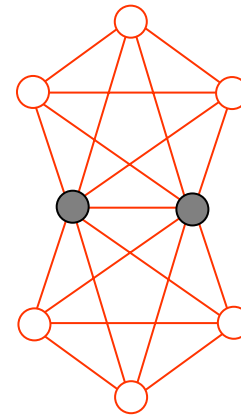
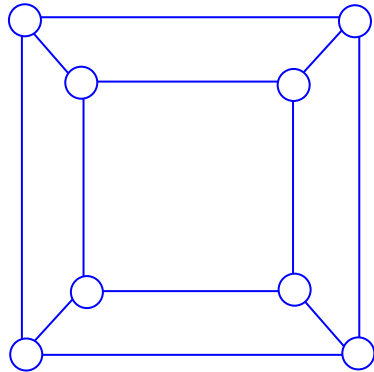
G k -connected: $|V(G)| > k$ and

G cannot be separated by $k-1$ nodes.

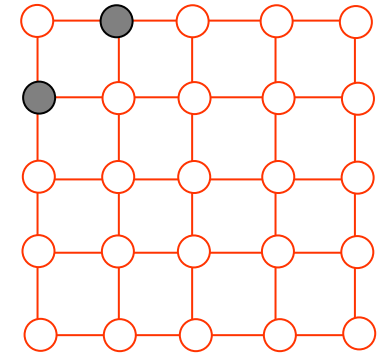
G is k -connected iff any two k -subsets are k -connected.



3-connected

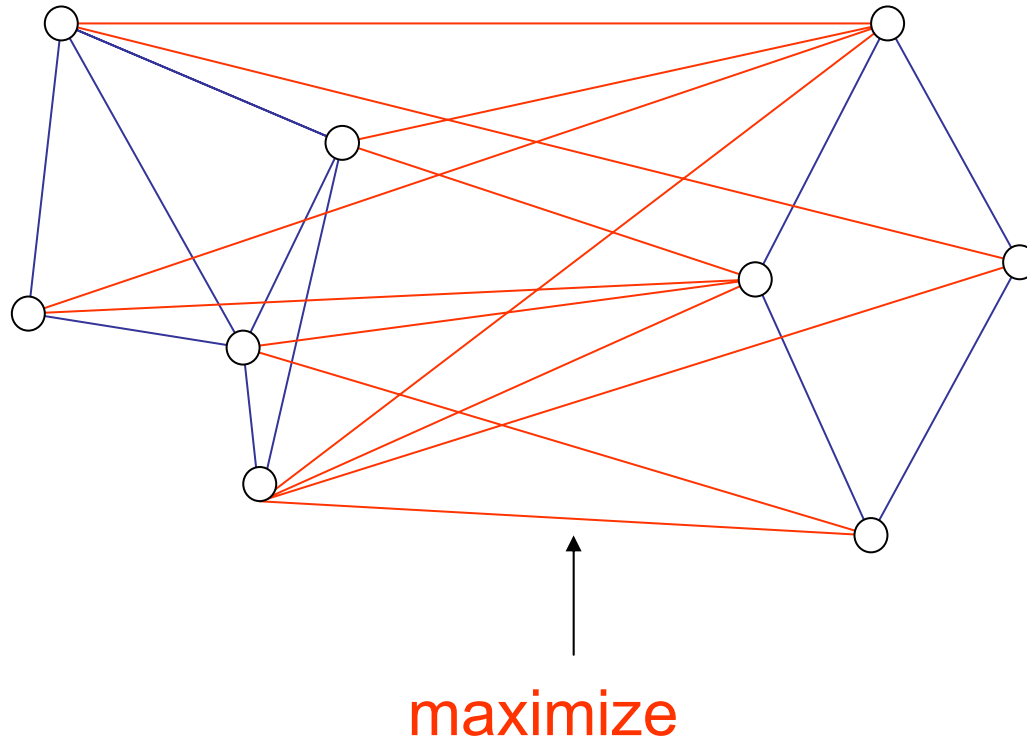


not 3-connected



Demo

The maximum cut problem



Applications: optimization, statistical mechanics...

Bad news: Max Cut is NP-hard

Approximations?

Easy with 50% error Erdős

Demo

Bad news: Max Cut is NP-hard

Approximations?

Easy with 50% error Erdős

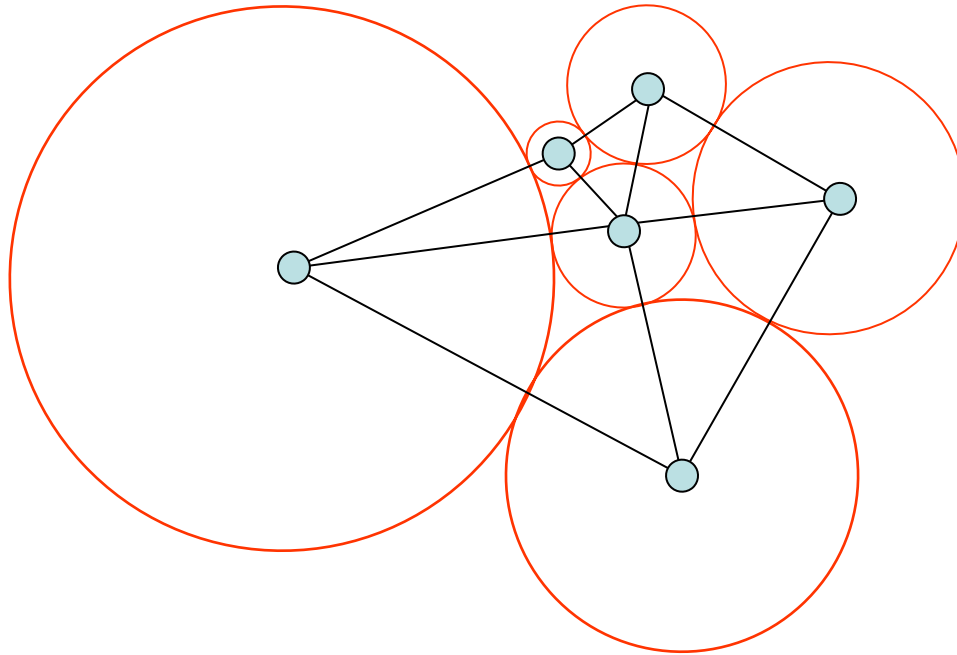
NP-hard with 6% error Hastad

Polynomial with 🕒 12% error

Goemans-Williamson

Demo

Coin representation Koebe (1936)

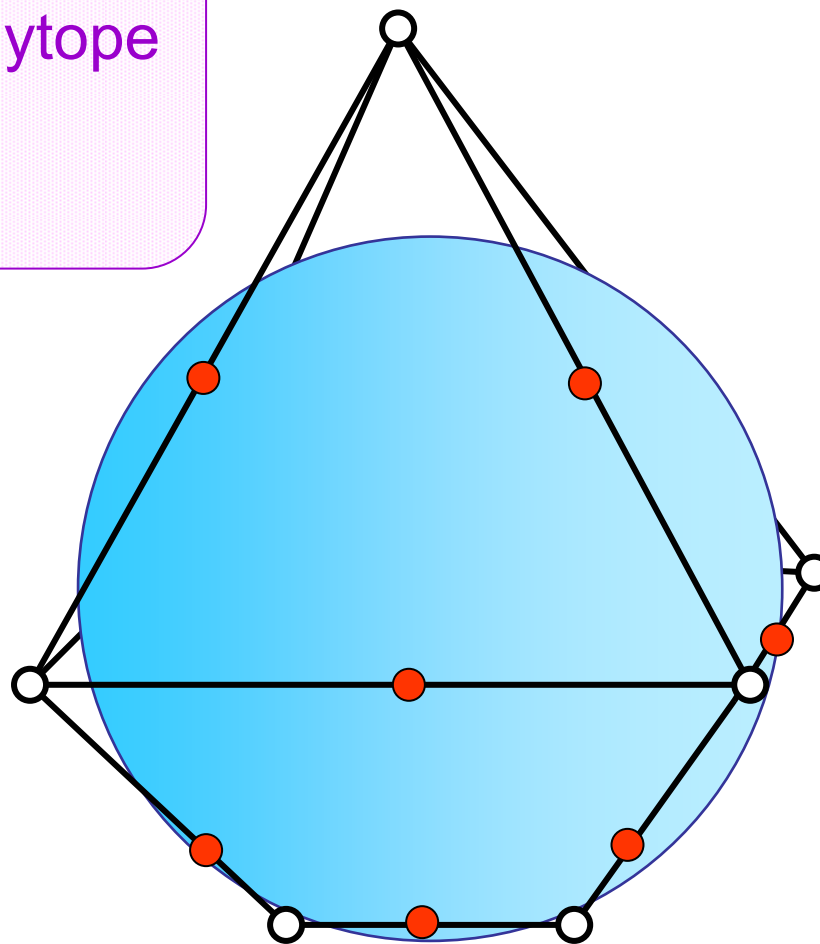


Every planar graph can be represented by touching circles

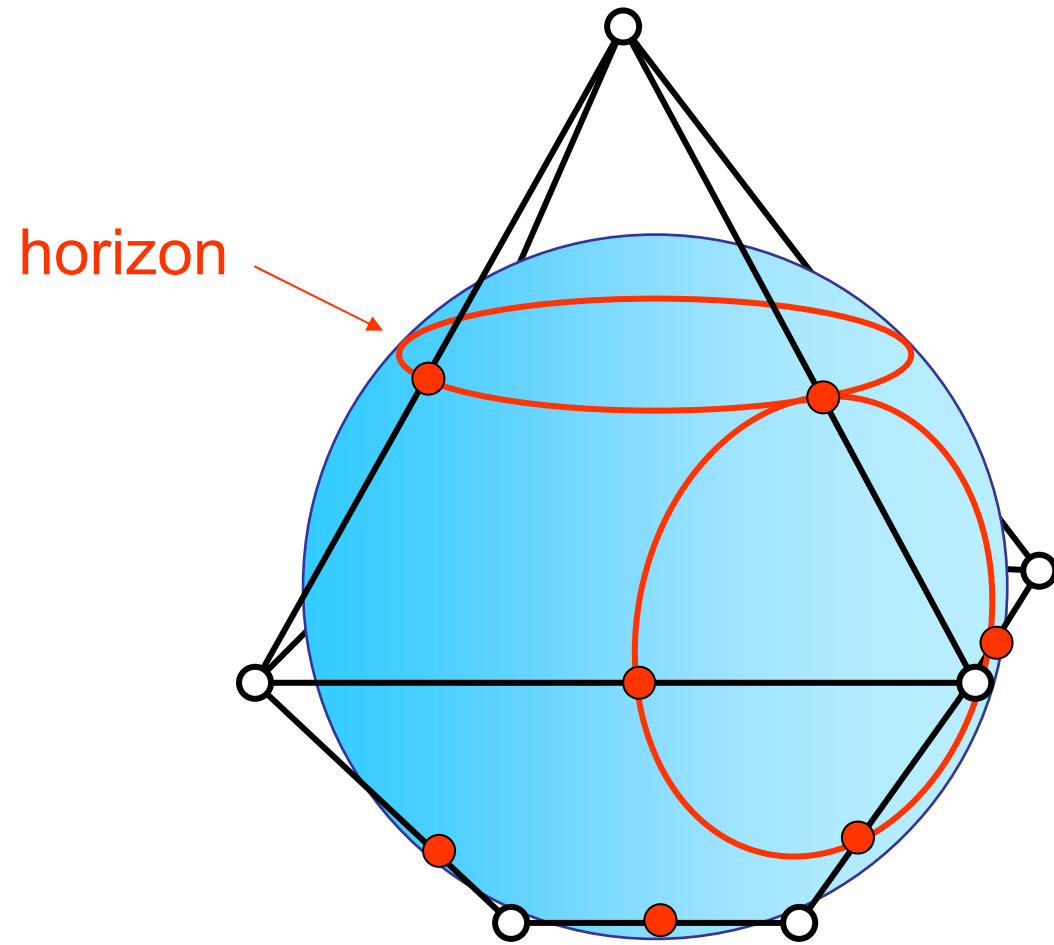
Representation by special polyhedra

Every 3-connected planar graph is the skeleton of a convex polytope such that every edge touches the unit sphere

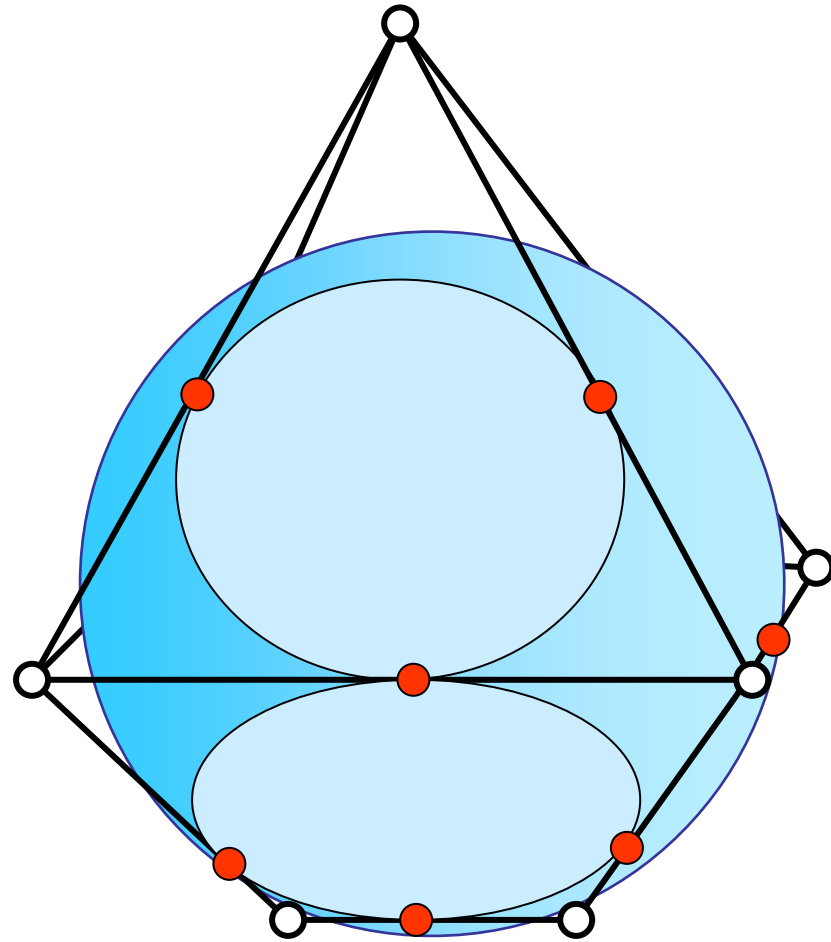
Koebe-Andreev-Thurston



From polyhedra to circles



From polyhedra to the polar



Planar Separator Theorem Lipton-Tarjan

Every planar graph G with n nodes contains an $S \subseteq V(G)$ with $|S| = O(\sqrt{n})$ such that

$$G \setminus S = G_1 \dot{\cup} G_2, \quad |V(G_1)|, |V(G_2)| \leq \frac{2n}{3} .$$

Proof (Miller and Thurston)

- Find Koebe representation on sphere;
- Modify so that center of gravity of circle centers is the center of sphere;
- Cut by random hyperplane through center of sphere.

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