Connectivity number

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Let $\mathcal{P} : ax+by+cz = 0$ be a plane through the origin. Discretization $P(a,b,c,\omega)$ of P can be analytically defined as the set of points satisfying the Diophantine inequalities $0 \leq ax + by + cz + \mu < \omega$. $\omega \in \mathbf{N}$ is the *arithmetic thickness* of the discrete plane $P(a,b,c,\omega)$. When $\omega = \max(|a|,|b|,|c|)$, $P(a,b,c,\omega)$ is called *naive.* We will denote such a plane $P(a,b,c)$, for short. The naive plane is equivalent to the classically defined discrete planes (see, e.g., [2]; see also [3] for a parallel with discrete lines). We assume that the coefficients a,b,c satisfy the conditions

$$0 < a < b < c \text{ and } gcd(a,b,c) = 1. \tag{1}$$

We will also suppose that the corresponding Euclidean plane P and the coordinate plane Oxy intersect at an angle θ such that $0 \leq \theta \leq \arctan\sqrt{2}$. Because of the well-known symmetry of the discrete space, the above conditions do not appear as restriction of the generality.

Now we give the following basic definition.

Definition 1 *Consider the function $\Omega : Z^3 \mapsto Z_+$ defined as follows:*

$$\Omega(a,b,c) = \max\{\ \omega\ :\ \text{the discrete plane } P(a,b,c,\omega) \text{ is disconnected }\}.$$

Thus $\omega = \Omega(a,b,c)+1$ is the least integer for which the discrete plane $P(a,b,c,\omega)$ is connected. For a particular choice of $a, b,$ and c, we call $\Omega(a,b,c)$ the **connectivity number** *relative to the class of discrete planes $\mathcal{C}(a,b,c) = \{\ P(a,b,c,\omega)\ :\ \omega = 0,1,2,\ldots\ \}$, or connectivity number of $P(a,b,c)$, for short.*

Note that the connectivity number is defined for arbitrary integer $a,b,$ and c, not necessarily satisfying conditions (1). Thus, for instance, we have $\Omega(a,b,c) = \Omega(b,a,c)$.

For inputs (a,b,c) satisfying $c \in [b, 2b-a] \cup [2b+a, +\infty)$ (Case 1) the solution is given in an explicit form by the following theorem [1].

Theorem 1

$$\Omega(a,b,c) = \begin{cases} c - a - b + gcd(a,b) - 1, & when\ c \geq 2b + a \\ b - a + gcd(a, c-b) - 1, & when\ c \leq 2b - a \end{cases}.$$

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Note that the computation of the above explicit solution requires $\Theta(\log b)$ operations, since this is the complexity of computing the greatest common divisor of two integers a and b.

For inputs with $c \in (2b - a, 2b + a)$ (Case 2) the solution can be found algorithmically in $O(a \log b)$ time [1].

Question 1: What is the optimal time to compute $\Omega(a, b, c)$ in Case 2? In particular, is it possible to compute $\Omega(a, b, c)$ in $O(\log b)$ time?

Question 2: Is it possible in Case 2 to explicitly express $\Omega(a, b, c)$ by a formula involving the given coefficients and elementary analytical or number-theoretical functions of them?

References

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