Multigrid Convergence of Discrete Differential Estimators: Discrete Tangent and 2D Length Estimation

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1 Introduction

In classical mathematics, differential estimators such that the tangent computation, the curvature calculus or the area estimation, are clearly defined and their properties are well known but when we want to apply this calculus on discrete data (2D or 3D discrete images), two different approaches are possible: we can first change the model of the data and put them into the classical continuous space by using interpolations or parameterizations of mathematical objects (B-splines, quadratic surfaces) on which the continuous curvature can be easily computed. Otherwise, we can try to express discrete curvature definitions and properties, and make sure that these new definitions are coherent with the continuous ones.

In the first approach, we have two main problems: the first one is that there exists a great number of parameterization algorithms in which some parameters have to be set according to the inputs. In order to provide a given accuracy, we have to reduce the input area and thus to limit our method. The second problem is that these algorithms have got a prohibitif computational time when we use large input data such as in medical imaging.

In a discrete approach, we first define discrete object such that Digital Straight Lines and Segments (DSL and DSS for short) or digital planes and we define discrete version of estimators based on such objects. Just to fix ideas, Discrete Tangent at a point of a discrete curve can defined as the longest DSS centered on the point.

This research report details several multigrid convergence proofs of discrete differential estimators. Note that this article is not self contained since it doesn't go into details about the digital geometry.

2 Multigrid digitization and multigrid convergence

Our studies on multigrid convergence require digitizations of planar Jordan curves up to a given grid resolution: we assume an orthogonal grid with grid constant $0 < \theta \leq 1$ in the Euclidean plane $\mathbb{R}^2$, *i.e.* θ is the uniform spacing between grid points parallel to one of the coordinate axes. Let $r = 1/\theta$ be the *grid resolution*, and the *r-grid* $\mathbb{Z}_r^2$ has resolution r, defined by *r-points* whose coordinates are $(\theta \cdot i, \theta \cdot j)$, with $i, j \in \mathbb{Z}$.

Now, we consider a Jordan curve $\gamma : [0,1] \to \mathbb{R}^2$, bounding a set S. Let $D_r(\gamma)$ be a *r-digitization* of γ in $\mathbb{Z}_r^2$, defined by *r-grid-intersection digitization*, or by a digital border in $\mathbb{Z}_r^2$. Common models are Gauss digitization (i.e. union of all r-grid squares with centroid in S), and inner or outer Jordan digitization (i.e. union of all r-grid squares contained in the interior of S, or having a non-empty intersection with S). See, e.g., [KY00] for details.

We denote by $\mathcal{F}(\gamma) \in \mathbb{R}$ a property of curve γ, which is the length $l(\gamma)$ of γ in this article. We denote by $\mathcal{E}$ an estimated feature. Assume that $\mathcal{E}$ is defined for digitizations $D_r(\gamma)$, for $r > 0$. The estimated feature $\mathcal{E}$ is said to be *multigrid convergent* iff $\mathcal{E}(D_r(\gamma))$ converges to $\mathcal{F}(\gamma)$, for $r \to \infty$. More formally:

$$|\mathcal{E}(D_r(\gamma)) - \mathcal{F}(\gamma)| \leq \epsilon(r)$$

with $\lim_{r\to\infty} \epsilon(r) = 0$. The order $O(1/\epsilon(r))$ denotes the *speed* of this convergence. Multigrid convergency of estimated features is a standard constraint in numerical mathematics for discrete versions of 'continuous' features.

3 Discrete normal vector field in 2D

The discrete tangent on a discrete curve was proposed by Vialard *et al.* in [Via96] and based on classical DSS (see figure 1):

Definition 1 (discrete tangent) *The discrete tangent at a point p of a discrete curve is the longest recognized DSS centered at p*

Note that this definition holds whatever the discrete curve type: we can define discrete tangent on 8-curves and also on cellular based boundary, we just have to consider the appropriated DSS algorithm.

Classical DSS recognition algorithms can be used directly but the tangent computation at each point of the curve becomes in $O(n^2)$. However, we use an optimization proposed by Feschet *et al.* [FT99] to compute all tangents in linear time.

Based on this calculus, we define a discrete normal vector at a point of a discrete curve:

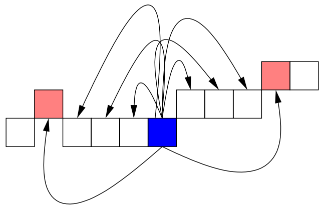

Figure 1: An example of a discrete tangent with symmetric DSS tests at a point of a discrete curve, the gray pixels end the tangent.

Definition 2 (discrete normal vector) *A discrete normal vector at a point of a discrete curve is defined by the orthogonal vector of the discrete tangent defined above.*

In the following, we prove that discrete normal vectors are multigrid convergent in direction, for an algorithm almost equivalent to DSS (the difference is discussed later and is necessary in order to prove the third hypothesis of the following theorem).

Theorem 1 *The discrete normal vectors defined above are multigrid convergent in direction.*

In the sequel we assume the following:

1. $\Gamma : [0,1] \rightarrow \mathbb{R}^2$ denotes the underlying continuous curve whose curvature is bounded by $C = \frac{1}{R}$.
2. At a point p_i of Γ we define a tube T_i centered in p_i of diameter θ and lengths $L(T_i)$ such that Γ belongs to the tube. The θ-enlargement of a tube is the union of this tube and two parts as described in figure 2 (for negative values of θ, the notion of enlargement is similarly defined).
3. $L(T_i) \geq K\sqrt{\frac{\theta}{C}}$, for any $i < N$.

This definition of centered tubes is geometrical explanation of the discrete tangent definition.

The begin and end of a tube are the intersection of the axis with the lateral faces (extremities). The pertinence of the last hypothesis is discussed later in the case of particular discretizations. Moreover, we assume that θ is small enough to avoid pathological cases such as half-turns in a tube ($\frac{\theta}{2} < R$). The goal is a bound on the error, linear in θ. Notice that the hypothesis (2) implies that Γ gets out of the $(-\theta)$-enlargements of the T_i's through lateral faces.

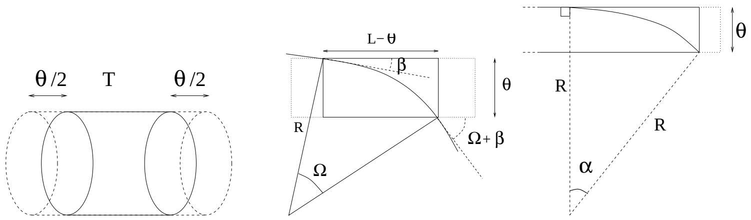

Figure 2: *Left*: θ-enlargement of tube T. *Middle*: schema for proof of lemma 1: L is the tube length, the dashed line is the tube contour, the solid line is the $(-\theta)$-enlargement of the tube. *Right*: case L large in Lemma 1

Lemma 1 *The angle between* Γ[1] *and the axis of* T_i *in the* $(-\theta)$*-enlargement of* T_i *is lower than* $B = \arcsin(\frac{\theta}{L-\theta}) + L(T_i)/R$*. If* α *such that* $\tan\alpha = \sqrt{\frac{\theta}{2R-\theta}}$ *verifies* $R\sin(\alpha) \leq L(T_i)$*, then the angle is bounded by* $B = \alpha$*.*

Proof 1 *See figure 2: this is the case of largest negative derivative for a projection on a plane including the direction of the tube. The following holds:*

$$\Omega \leq L/R \quad (1)$$

$$\sin\beta \leq \frac{\theta}{L-\theta} \quad (2)$$

This yields the first result. The particular case in which $R\sin(\alpha) \leq L(T_i)$ *for* $\tan\alpha \leq \sqrt{\frac{\theta}{2R-\theta}}$ *is illustrated in figure 2-Right; in this case, L is large, and the derivative must be small enough to keep* Γ *in the enlargement of the tube. This proves that the angle is bounded by* $arc\tan(\sqrt{\frac{\theta}{2R-\theta}})$*.*$\square$

The Lemma shows that the maximum angular error between the normal vector defined by the tube and the normal vector of Γ in T is bounded by $O(\sqrt{\theta C})$ under the above assumptions. The error tends to zero when $r = 1/\theta$ tends to ∞. Hence, we prove the theorem 1.

We now have to study the hypothesis according to which $L(T_i) \geq K\sqrt{\delta/C}$, for any $i < N$.

This hypothesis is true whenever the discretization and polygonalization verify that any curve which lies in a tube of radius ϵr with the resolution r is included in a segment (this can easily be proved by considering the fact that for bounding curvature C, the minimal length of Γ before ϵr-deviation from the tangent is $\Theta(\sqrt{\frac{\epsilon r}{C}})$). The classical DSS algorithm does not verify this property for some pathological cases. It is likely that adding some reasonnable constraints on the curve could remove this condition. Further details can be found in [CDRT01].

[1] This angle is the maximal one between a tangent of Γ and the tube axis.

4 Discrete Integration of a Normal Vector Field

Based on this discrete tangent calculus, we can define an algorithm to estimate the length of the curve.

Let $\vec{n} : [0, 1] \to \mathcal{E}^2$ denotes the normal vector field associated to the Euclidean curve $\mathcal{C}$. The length of $\mathcal{C}$ can be expressed:

$$l(\mathcal{C}) = \int_{\mathcal{C}} \vec{n}(s) ds \tag{3}$$

The main idea of tangent based approach is to consider estimated normals at each point the discrete curve and pixel elements as an estimation of ds.

Hence, we consider a cellular based discrete curve [Kov89]. The Vialard's algorithm computes the discrete tangent and thus the discrete normal vector at each $0-$cell of the curve. We define the normal vector $\tilde{n}$ associated to each $1-$cell as the mean value vector of the neighbor $0-$cell. We also define an elementary normal vector n_{el} to a $1-$cell as the unit vector orthogonal to the cell. Hence, the discrete version of eq. 3 is:

$$l_{TangInt}(D_r(\mathcal{C})) = \sum_{s \in \mathcal{S}} \tilde{n}(s).n_{el}(s) \tag{4}$$

where '.' denotes the scalar product and $\mathcal{S}$ the set of $1-$cell of $D_r(\mathcal{C})$. The main idea of this approach is to compute the contribution of each $1-$cell to the global length by projecting the $1-$cell according to the normal vector's direction.

We call this process *a discrete integration of a normal vector field.*

In the following, we prove that the result of the normal field integration converges to the Euclidean measure if the normal vector estimation is multigrid convergent. We present a more general proof that shows the multigrid convergence of the surface area estimation using same idea: we first estimate the normal vectors at each point of a discrete surface and we estimate the area of the surface as the discrete integration of a normal vector field. The proof of the length estimator can be easily deduced.

4.1 Surface area estimation

We consider a compact surface $\mathcal{S}$ in $\mathcal{E}^3$ with continuous derivative. We consider a digitization function f_r which is an application from $S \in \mathbb{R}^d$ to $\mathbb{R}^d$ and such that df_r converges weakly to the identity (*i.e.* for any continuous function g we have $\int_{\mathcal{S}} g.df_r(ds) \to \int_{\mathcal{S}} gds$). We also need that df_r and f_r are bounded for r small enough, and f must converge (in the usual sense) to the identity on $\mathcal{S}$.

In other words, f_r is a digitization function such that $f_r(\mathcal{S})$ converges to $\mathcal{S}$ when r converges to zero. The weak convergence property intuitivly just means that we can evaluate integrals using the discretization. The interest of this formalism is that simple remarks using powerfull technical lemmas from functional analysis yield interesting results.

Notice that this definition of a digitization function is not usual, but any classical digitization schemes such as the Grid Intersect Quantization (GIQ for short), Background Boundary Quantization (BBQ for short) or Object Boundary Quantization (OBQ for short) (see [JK97] for a survey on digitization schemes) lead to the existence of such a function (not uniquely defined). This point will be detailed at the end of this section.

A solution for evaluating a surface in a continuous case consists in using $\int_{\mathcal{S}} \overrightarrow{n}(s)\overrightarrow{ds}$, with $\overrightarrow{n}(s)$ the normal to $\mathcal{S}$ in s.

The discrete approximation described above consists in using $\int_{f_r(\mathcal{S})} \overrightarrow{n}^*(s')\overrightarrow{ds'}$ (notice that whenever we use an integral notation, this is a finite sum as all elements are constant on a finite number of areas in usual digitization), with $\overrightarrow{n}^*$ an evaluation of the normal. We only assume that $\overrightarrow{n}^*(s')$ converges uniformly to $\overrightarrow{n}(s)$ as $d(s,s') \rightarrow 0$. Precisely:

$$\lim_{\epsilon \rightarrow 0} \sup_{d(s,s') \leq \epsilon} |\overrightarrow{n}^*(s') - \overrightarrow{n}(s)| = 0$$

Notice that the hypothesis can be rewritten as a simple convergence condition, as we work on continuous function on compact sets. Anyway, usual approximation results will directly lead to this formula.

The discrete integral is then equal to:

$$\int_S \overrightarrow{n}^*(f_r(s))df_r(\overrightarrow{ds})$$

And we define:

$$\Delta = |\int_S \overrightarrow{n}^*(f_r(s))df_r(\overrightarrow{ds}) - \int_S \overrightarrow{n}(s)\overrightarrow{ds}|$$

if r is small enough to ensure that $|\overrightarrow{n}^*(f_r(s)) - \overrightarrow{n}(s)| < \epsilon$, then (thanks to the finiteness of df_r)

$$\Delta = O(\epsilon) + |\int_S \overrightarrow{n}(s)df_r(\overrightarrow{ds}) - \int_S \overrightarrow{n}(s)\overrightarrow{ds}|$$

Thanks to the smoothness of $\overrightarrow{n}(s)$ and to the weak convergence of df_r, the term $|.|$ converges to 0 as $r \rightarrow 0$.

Hence, we obtain the following theorem:

Theorem 2 (Consistence of the discrete evaluation of the surface area in any dimension) *If df_r converges weakly, as the precision increases, to the identity, with f_r the digitization, and if $\overrightarrow{n}'(f_r(s))$ converges uniformly to $\overrightarrow{n}(s)$, then the discrete integral converges to the surface area and the convergence speed is $O(\epsilon)$.*

Corollary 1 *The surface area estimator based on a discrete integration of a normal vector field is multigrid convergent if and only if the normal vector field is convergent in direction.*

We restrict our attention to the case in which there's no self-intersection in $\mathcal{S}$. Other cases can be treated as well but are not interesting in many cases and do not lead to an interesting developement of the proof.

For r sufficiently small (such that there is no "turn back" in the discretization), we define $f(s) = s'$ such that $s' \in s + \mathbb{R}\overrightarrow{n}(s)$ and $d(s, s')$ is minimal. The existence, uniqueness, continuity, derivability of s' are clear almost everywhere except if S' is orthogonal to S in a neighborhood of s. All the work consists in proving that this (almost) never occurs. Fortunately, this occurs for $\overrightarrow{n}(s) \in E$, with E of measure 0, set of (sic!) normals to normals in discretizations. E has measure 0 as a finite union of sets of measure 0 (there are a finite number of possible normals in a discretization, hence a finite number of hyperplanes of normals to normals of a discretization).

The important point is now that this does not conclude the proof. This proves that for a given s, the probability of having $\overrightarrow{n}(s) \in E$ is 0; we need this on any neighbourhood of $s \in S$.

Consider now $d(s)$ for $s \in \mathcal{S}$ the dimension of the set of $\overrightarrow{n}(s'')$ for s'' in the neighbourhood of s. d is an integer-valued function, constant except for a set of measure 0. Define S_i for $i \in I$ the different maximal connex subsets of S on which d is different of the embedding dimension. Necessarily I is countable, as each S_i has a positive measure, and a finite sum is necessarily countable (easily derived from the fact that the number of S_i of measure larger than a given $t > 0$ is necessarily finite). Hence, the sum (over $i \in I$) of the probabilities of $\overrightarrow{n}(s) \in E$ for $s \in S_i$ (the random variable being the choice of the angles defining the grid) is a countable sum of probabilities equal to 0; hence this probability is 0.

People who do not like arguments based upon sets of measure 0 can indeed find other proofs, less elegant than the above argument, which according to us has the advantage of providing such a function f.

The convergence of f_r towards the identity being clear, we have to verify the weak convergence of df_r to the identity. This is indeed a simple consequence of the fact that f_r converges almost everywhere to the identity, thanks to the classical result stating that the almost sure convergence implies the weak convergence of the derivative. Other proofs based upon the Green-Ostrogradsky theorem can be provided as well.

4.2 Length estimation

With theorem 1 and corollary 1, we can deduced the theorem:

Theorem 3 *The length estimator based on the discrete vector field integration of the discrete normal vector field defined above is multigrid convergent.*

5 Conclusion

In this article, we have presented several multigrid convergent proofs: the first one presents the convergence of a normal vector field on 2D discrete curve and the second one shows that we can integrate a normal vector field and obtain a multigrid convergent estimator under the assumption that the normal vector field converges. We finally prove that the length estimation of a 2D discrete curve with the proposed algorithm converges asymptotically to the Euclidean length of the underlying continuous curve.

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