Can 3-D Digital Topology be Based on Axiomatically Defined Digital Spaces?

T. Yung Kong
Department of Computer Science
Queens College
City University of New York
Flushing, NY 11367, U.S.A.
E-mail: ykong@euler.cs.qc.edu

June 22, 2001

For the 2-d and 3-d Cartesian grids, and commonly used non-Cartesian grids such as the 3-d face-centered and body-centered cubic grids, there are certain "good pairs" of adjacency relations $(\alpha, \overline{\alpha})$ on the grid points. For these pairs of relations $(\alpha, \overline{\alpha})$, many results of digital topology concerning a set of grid points and its complement have versions in which α-adjacency is used to define connectedness on the set and $\overline{\alpha}$-adjacency is used to define connectedness on its complement.

For example, (4,8) and (8,4) are good pairs of adjacency relations on $\mathbb{Z}^2$, but (4,4) and (8,8) are not.[1] Thus Rosenfeld's digital Jordan curve theorem [1] is valid when one of 4- and 8-adjacency is used to define the sense in which a digital simple closed curve is connected and the other of the two adjacency relations is used to define connected components of the digital curve's complement. But the theorem is not valid if the same one of 4- or 8-adjacency is used for both purposes.

[1] We use the convention that if α is an irreflexive symmetric binary relation on the set G of grid points of a Cartesian or non-Cartesian grid, then α is referred to as the *k-adjacency* relation on G, and is denoted by the positive integer k, if for all $p \in G$ the set $\{q \in G \mid p \ \alpha \ q\}$ contains exactly k points and those points are all strictly closer to p (in Euclidean distance) than any other point of $G \setminus \{p\}$ is.

In three dimensions, (6,26), (26,6), (6,18), (18,6) are good pairs of adjacency relations on $\mathbb{Z}^3$, $(12,12)$, $(12,18)$ and $(18,12)$ are good pairs of adjacency relations on the points of a 3-d face-centered cubic grid (e.g., $\{(x,y,z) \in \mathbb{Z}^3 \mid x+y+z \equiv 0 \pmod 2\}$) and (14,14) is a good pair on the points of a 3-d body-centered cubic grid (e.g., $\{(x,y,z) \in \mathbb{Z}^3 \mid x \equiv y \equiv z \pmod 2\}$).

At present, results of digital topology are usually proved for one good pair of adjacency relations at a time, and the details of the proof may be significantly different for different good pairs. This is unsatisfactory in the 3-d case, because many different good pairs seem to deserve consideration. Even if we restrict our attention to the good pairs of adjacency relations mentioned in the previous paragraph, a result such as a digital Jordan surface theorem would be expected to have eight different versions.

Problem: Construct a simplest possible theory that gives an axiomatic definition of "well-behaved 3-d digital spaces" and allows many results of 3-d digital topology to be proved simultaneously for all such spaces. The axioms should be general enough to admit digital spaces that correspond to the 3-d grids and good pairs of adjacency relations mentioned above.

Reference

[1] A. Rosenfeld, Arcs and curves in digital pictures, *J. ACM* **20**, 1973, 81–87.