Counting the number of k-dimensional grid cells of $\mathbb{Z}^d$ that intersects a polytope

Jacques-Olivier Lachaud

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Counting the number of lattice points of $\mathbb{Z}^d$ that a lattice polytope $\mathcal{P}$ intersects is a famous problem in discrete mathematics, and is at the basis of linear integer programming. A fruitful way of addressing this problem is to generalize this counting to dilation $t\mathcal{P}$ of $\mathcal{P}$, for some parameter t. We thus write:

$$\mathcal{L}(\mathcal{P}, t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right). \tag{1}$$

Ehrhart showed in 1962 that $\mathcal{L}$ is a rational polynomial of degree d in t. Note that this result holds for arbitrary lattice in $\mathbb{R}^d$, not only for the integer lattice $\mathbb{Z}^d$.

Now let us consider the (cubical) cell complex $\mathcal{C}^d$ induced by the lattice $\mathbb{Z}^d$, such that its 0-cells are the points of $\mathbb{Z}^d$, its 1-cells are the (closed) unit segments joining two 0-cells at distance 1, its 2-cells are the (closed) unit squares formed by these segments, $\ldots$, and its d-cells are the d-dimensional unit hypercubes with vertices in $\mathbb{Z}^d$. We further denote $\mathcal{C}^d_k$ the set of its k-cells.

Open Problem 1 *We count the number of intersections of $\mathcal{P}$ with the cell complex $\mathcal{C}^d$. Let k be an integer with $0 \leq k \leq d$.*

$$\mathcal{L}_k(\mathcal{P}, t) := \#\left(t\mathcal{P} \cap \mathcal{C}^d_k\right). \tag{2}$$

- *We have $\mathcal{L}_0(\mathcal{P}, t) = \mathcal{L}(\mathcal{P}, t)$.*
- *Question 1: for $0 < k \leq d$, is $\mathcal{L}_k(\mathcal{P}, t)$ still a rational polynomial of degree d in t ?*
- *Question 2: for $0 < k \leq d$, do we keep the very nice Ehrhart-MacDonald reciprocity ? This could be formulated as*

$$\mathcal{L}_k(\mathrm{Int}\left(\mathcal{P}\right), t) = (-1)^d \mathcal{L}_k(\mathcal{P}, -t)? \tag{3}$$

- *Question 3: If $\mathcal{L}_k(\mathcal{P}, t)$ is a polynomial, is there an interpretation of its coefficients ? For instance, for $k = 0$, it is known that: Coefficient of monomial t^d is the volume of the polytope (divided by the size of each volume of the lattice, so 1 for $\mathbb{Z}^d$). Coefficient of monomial t^0 is the Euler characteristic of the polytope, i.e. 1.*

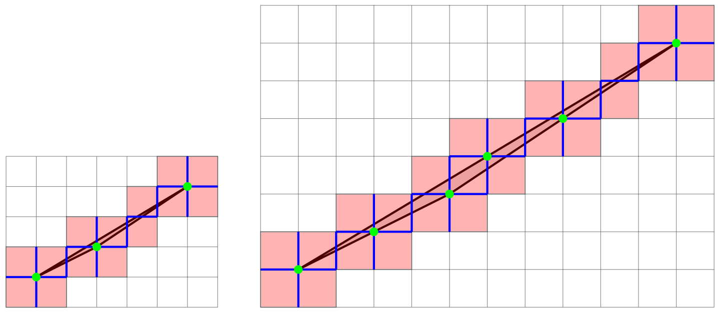

Figure 1: Left: $\mathcal{L}_0(\mathcal{P},1)=3$, $\mathcal{L}_1(\mathcal{P},1)=16$, $\mathcal{L}_2(\mathcal{P},1)=14$. Right: $\mathcal{L}_0(\mathcal{P},2)=6$, $\mathcal{L}_1(\mathcal{P},2)=30$, $\mathcal{L}_2(\mathcal{P},2)=25$.

- *We must probably have:* $\sum_{k=0}^{d}\mathcal{L}_k(\mathcal{P},t)=(-1)^d$*, since the cubical complex that is the intersection of* $\mathcal{C}^d$ *with* $t\mathcal{P}$ *is open and the euler characteristic of its dual must be* 1*.*