Minimal Non-Simple and Minimal Non-Cosimple Sets in Binary Images on Cell Complexes

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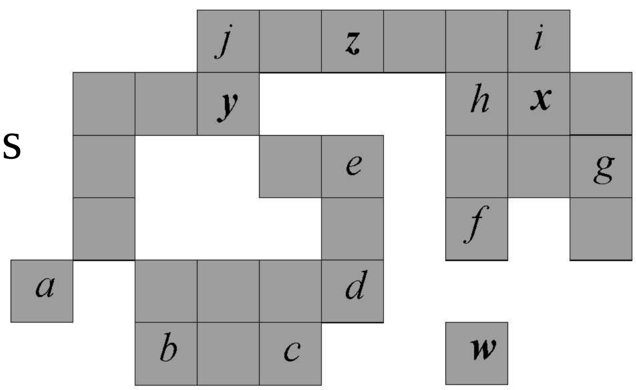
Background: Ronse's Theorems

In the 1960's Rosenfeld introduced the concepts of 8-***simple*** and 4-***simple*** 1's in binary images on a 2D Cartesian grid.

An 8-simple 1 is a ***non-8-isolated 4-border*** 1 that can be changed to 0 ***without*** splitting any 8-connected object, and also ***without*** merging any 4-connected hole with the background or with another such hole.

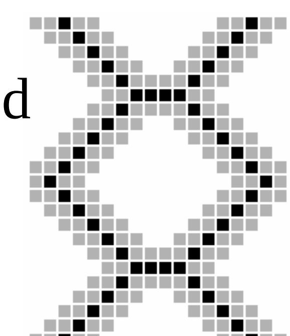
$a, b, ..., j$ are some 8-simple 1's of a 2D binary image (whose 1's are the gray squares).

w, *x*, *y*, *z* are some *non*-8-simple 1's.



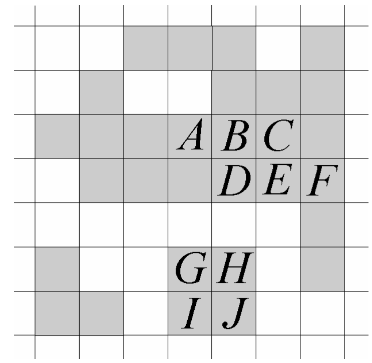
4-simple 1's are analogous, but with the roles of "4-" and "8-" interchanged.

An important application of these concepts is to the problem of establishing that proposed thinning algorithms "preserve topology".



[Thinning algorithms are used to reduce objects in binary images down to thin "skeletons".]

The concepts of **8-*deletable*** and **4-*deletable sets*** generalize the concepts of 8-simple and 4-simple 1's to finite sets of zero or more 1's. [*e.g.*, {*B*,*C*,*D*,*E*} *is* 8-*deletable*]



An **8-*deletable*** set can be defined as a (finite) set $\mathcal{D}$ of 1's such that, when the elements of $\mathcal{D}$ are changed to 0's, ***none of the following occurs***:

- an 8-connected object is split [*e.g.*, {*E*,*F*} *is* ***not*** 8-*deletable*]
- a 4-connected hole is merged with the background or merged with another such hole [{*A*,*B*,*D*} *is* ***not*** 8-*deletable*]
- an 8-connected object vanishes [{*G*,*H*,*I*, *J*} *is* ***not*** 8-*deletable*]
- a new 4-connected hole is created [{*B*, *C*} *is* ***not*** 8-*deletable*]

4-*deletable* sets are analogous—just switch "4" and "8".

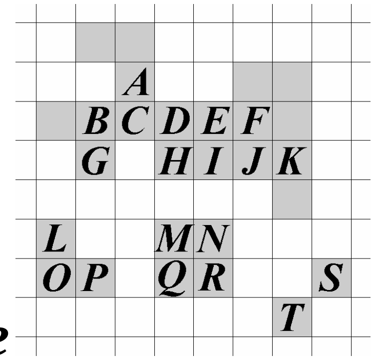
[**Note**: Ronse called these sets *strongly* 8-(4-)*deletable*.]

In the mid-1980's, Ronse proved the next two theorems, which provide the basis for a powerful method of establishing that a proposed parallel thinning algorithm is "8-topology-preserving" or "4-topology preserving".

[Here "8-(4-)topology-preserving" means: *For every possible input image* $\mathbb{I}$, *the set of* 1*'s of* $\mathbb{I}$ *that are changed to* 0 *by the algorithm is an* 8-(4-)*deletable set of* $\mathbb{I}$.]

A ***minimal non-*****8-(4-)*****deletable*** set of a binary image $\mathbb{I}$ is a set $\mathcal{D}$ of 1's of $\mathbb{I}$ such that:

1. Each proper subset of $\mathcal{D}$ is an 8-(4-)deletable set of $\mathbb{I}$.
2. $\mathcal{D}$ is ***not*** an 8-(4-)deletable set of 1's of $\mathbb{I}$.



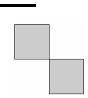
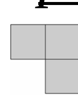
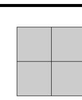
Example: The minimal non-8-deletable sets in the image on the right (■ = 1, □ = 0) are {*A*}, {*F*}, {*B*,*C*}, {*B*,*G*}, {*C*,*D*}, {*D*,*H*}, {*E*,*I*}, {*J*,*K*}, {*L*,*O*,*P*}, {*M*,*N*,*Q*,*R*}, {*S*,*T*}

We say that a given set $\mathcal{D}$ of pixels ***can be minimal non-*****8-(4-)*****deletable*** if there exists an image $\mathbb{I}$ such that $\mathcal{D}$ is a minimal non-8-(4-)deletable set of $\mathbb{I}$.

We say that a given set $\mathcal{D}$ of pixels ***can be minimal non-*****8-(4-)*****deletable as a proper subset of a component*** if there exists an image $\mathbb{I}$ such that $\mathcal{D}$ is a minimal non-8-(4-)deletable set of $\mathbb{I}$, and $\mathcal{D}$ is a *proper* subset of an 8-(4-)component of the 1's of $\mathbb{I}$.

Theorem 1 (Ronse, 1988) *A set $\mathcal{D}$ of pixels can be minimal non-*8*-deletable* ***if and only if*** *one of the following is true*:

(1) *$\mathcal{D}$ is a singleton set or a pair of* 4*-neighbors*.

(2) *$\mathcal{D}$ is isometric to*  ,  , *or*  .

*$\mathcal{D}$ can be minimal non-*8*-deletable as a proper subset of a component* ***if and only if*** *$\mathcal{D}$ satisfies* (1).

Recall:

We say that a given set $\mathcal{D}$ of pixels ***can be minimal non*-8-(4-)*deletable*** if there exists an image $\mathbb{I}$ such that $\mathcal{D}$ is a minimal non-8-(4-)deletable set of $\mathbb{I}$.

We say that a given set $\mathcal{D}$ of pixels ***can be minimal non*-8-(4-)*deletable as a proper subset of a component*** if there exists an image $\mathbb{I}$ such that $\mathcal{D}$ is a minimal non-8-(4-)deletable set of $\mathbb{I}$, and $\mathcal{D}$ is a *proper* subset of an 8-(4-)component of the 1's of $\mathbb{I}$.

Theorem 1 (Ronse, 1988) *A set $\mathcal{D}$ of pixels can be minimal non*-8-*deletable* ***if and only if*** *one of the following is true*:

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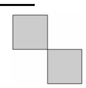
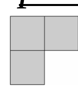
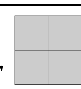
(2) $\mathcal{D}$ *is isometric to*  ,  , *or*  .

$\mathcal{D}$ *can be minimal non*-8-*deletable as a proper subset of a component* ***if and only if*** $\mathcal{D}$ *satisfies* (1).

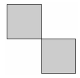
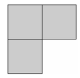
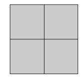
For minimal non-4-deletable sets, the analogous result is:

Theorem 2 (Ronse, 1988) *A set $\mathcal{D}$ of pixels can be minimal non*-4-*deletable* ***if and only if*** $\mathcal{D}$ *is a singleton set or a pair of* 8-*neighbors*. *In both cases, $\mathcal{D}$ can be minimal non*-4-*deletable as a proper subset of a component.*

[**Reference**: C. Ronse, Minimal test patterns for connectivity preservation in parallel thinning algorithms for binary digital images, *Discrete Applied Mathematics* **21**, 1988, 67–79.]

Theorem 1 (Ronse, 1988) *A set $\mathcal{D}$ of pixels can be minimal non-*8*-deletable* ***if and only if*** *one of the following is true*:

(1) *$\mathcal{D}$ is a singleton set or a pair of* 4*-neighbors*.

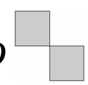
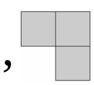
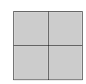
(2) *$\mathcal{D}$ is isometric to*  ,  , *or*  .

*$\mathcal{D}$ can be minimal non-*8*-deletable as a proper subset of a component* ***if and only if*** *$\mathcal{D}$ satisfies* (1) .

Theorem 2 (Ronse, 1988) *A set $\mathcal{D}$ of pixels can be minimal non-*4*-deletable* ***if and only if*** *$\mathcal{D}$ is a singleton set or a pair of* 8*-neighbors. In both cases, $\mathcal{D}$ can be minimal non-*4*-deletable as a proper subset of a component.*

Therefore, to establish that a parallel thinning algorithm **T** "preserves 8-topology", it suffices to show that:

The set of 1*'s which are changed to* 0 *at a single subiteration of* **T** *never includes the following*:

- *a singleton or pair of* 4*-neighbors that is a non-*8*-deletable set* *
- *an* 8*-component of the* 1*'s that is isometric to*  ,  , *or* 

Similarly, to establish that a parallel thinning algorithm **T** "preserves 4-topology", it suffices to show that:

The set of 1*'s which are changed to* 0 *at a single subiteration of* **T** *never includes the following*:

- *a singleton or pair of* 8*-neighbors that is a non-*4*-deletable set* *

* *of the image at the start of that subiteration*

Since the 1980's, analogs of Ronse's two theorems have been obtained for binary images on other grids:

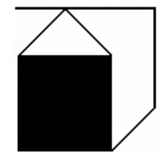
- 2D hexagonal grid (Hall)
  [*Topology and Its Applications* **46**, 1992, 199–217.]
- 3D Cartesian grid (Ma)
  [*CVGIP: Image Understanding* **59**, 1994, 328–39.]
- 3D face-centered cubic grid (Gau & Kong)
  [*International Journal of Pattern Recognition and Artificial Intelligence* **13**, 1999, 485–502.]
- 4D Cartesian grid (Gau & Kong)
  80-connectedness (on 1's): [*Graphical Models* **65**, 2003, 112–30.]
  8-connectedness: [in: R. Klette, J. Žunić (eds.), *Proc. IWCIA 2004*, 318–33.]

However, each grid has been dealt with separately, using arguments many of whose details are specific to that grid.

Our main results unify and extend this earlier work: We generalize the two theorems of Ronse to a very large class of grids of dimension ≤ 4 (a class that includes all of the above-mentioned grids).

The key idea is to base our arguments on a simple fact about intersections and unions of contractible polyhedra in 3-space.

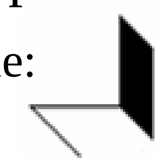
Background: Contractible Polyhedra in $\mathbb{R}^3$



A ***polyhedron*** is a set that is expressible as the union of a *finite* collection of simplexes (which may contain simplexes of different dimensionalities).

A polyhedron P is said to be ***contractible*** if P can be "continuously deformed over itself" down to one point.

Examples: 1. any convex polytope. 2. a nonconvex example:
3. a set consisting of a single point.



A polyhedron P in $\mathbb{R}^3$ is contractible ***if and only if***

- P is *nonempty*, *connected*, and *simply connected*, and
- $(\mathbb{R}^3 \setminus P)$ *is connected*.

Informally: A polyhedron $P \neq \emptyset$ in $\mathbb{R}^3$ is *contractible* ***iff*** P is connected, has no internal cavities, and has no holes.

Here is a *computationally* convenient characterization:

A polyhedron P in $\mathbb{R}^3$ is contractible ***iff***

- P is *connected*, and
- $(\mathbb{R}^3 \setminus P)$ *is connected*, and
- $\chi(P) = 1$. [$\chi(P)$ denotes the ***Euler characteristic*** of P.]

Unions and Intersections of Contractible Polyhedra

Our work is based on the following topological fact:

Key Fact: *Let P_1 and P_2 be polyhedra in* $\mathbb{R}^3$. *Then* *<u>any two of the following imply the third</u>:*

1. *Each of P_1 and P_2 is contractible.*
2. *$P_1 \cup P_2$ is contractible.*
3. *$P_1 \cap P_2$ is contractible.*

[This follows from the "*P and* ($\mathbb{R}^3 \setminus P$) *are connected*, *and* $\chi(P) = 1$" characterization of contractible polyhedra *P* in $\mathbb{R}^3$, and results of algebraic topology (e.g., the reduced Mayer-Vietoris sequence).]

Although $1 \wedge 3 \Rightarrow 2$ and $2 \wedge 3 \Rightarrow 1$ are true even without the hypothesis that P_1 and P_2 are in $\mathbb{R}^3$, $1 \wedge 2 \Rightarrow 3$ ***would be false without that hypothesis***.

This is one of the (two) places where our arguments depend on the assumption that our binary images are defined on complexes of dimension ≤ 4.

Background: Boundary Faces and Schlegel Diagrams

Notation: If Q is any nD convex polytope, then

$$\textbf{bdryfaces}(Q) \stackrel{\text{def}}{=} \textit{the set of all the lower-dimensional faces of } Q.$$

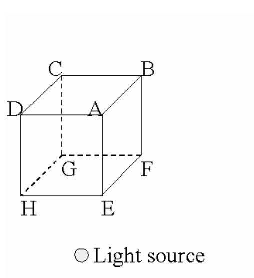
$$\textbf{faces}(Q) \stackrel{\text{def}}{=} \{Q\} \cup \textbf{bdryfaces}(Q)$$

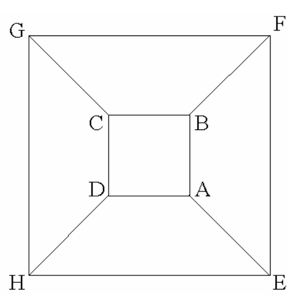
Example: If Q is a cube, then **faces**(Q) *has* **27** *elements* and **bdryfaces**(Q) *has* **26** *elements*—because Q has 8 vertices, 12 edges, and 6 2D faces.

A ***Schlegel diagram*** represents **bdryfaces**(Q), in a topologically faithful way, as a *collection of cells whose union is* $\mathbb{R}^{n-1} \cup \{\infty\}$.

Example

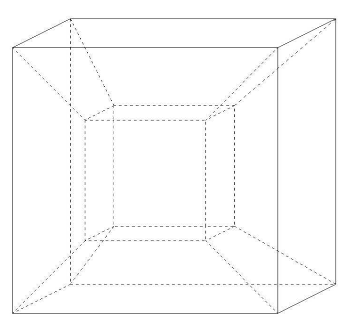
A cube Q





A (2D) Schlegel diagram of **bdryfaces**(Q) — note that the "outside region" represents the "bottom" 2D face of **bdryfaces**(Q).

If Q is a 4D hypercube, then a Schlegel diagram of **bdryfaces**(Q) is **3-*dimensional*** and looks *like this*:



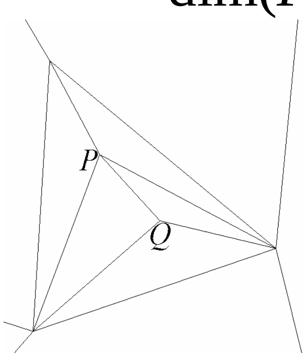
*n*D Xel Complexes; *m*-Xels

Our main results are proved for binary images on the grid cells of 2D, 3D, and 4D *xel complexes*.

For simplicity, we will only define *convex* xel complexes in this talk.

An *n*D ***convex xel complex*** **K** is a set of convex polytopes that satisfies the following conditions:

1. $\bigcup \mathbf{K} = \mathbb{R}^n$ and **K** is locally finite.
2. If $Q \in \mathbf{K}$ then $\mathbf{faces}(Q) \subseteq \mathbf{K}$.
3. If $P, Q \in \mathbf{K}$ and $P \cap Q \neq \emptyset$, then $P \cap Q \in \mathbf{faces}(P) \cap \mathbf{faces}(Q)$.
4. If $P, Q \in \mathbf{K}$ and $P \cap Q = \emptyset$, then there exist $P', Q' \in \mathbf{K}$ such that $P \in \mathbf{faces}(P')$, $Q \in \mathbf{faces}(Q')$, $\dim(P') = \dim(Q') = n$, and $P' \cap Q' = \emptyset$.



Condition 4 ***excludes*** complexes that include configurations such as the one on the left. Here $n = 2$.

Each element *P* of **K** is called a ***xel*** (of the complex **K**). If $P \in \mathbf{K}$ and $\dim(P) = m$, then *P* is called an ***m-xel*** (of **K**).

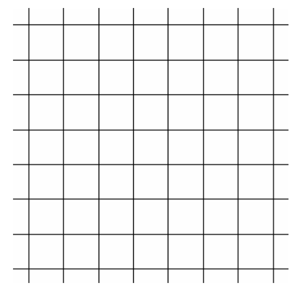
Notation: If **K** is an *n*D xel complex, then we write $\mathcal{G}$(**K**) to denote the *set of all n-xels of* **K**.

Each element of $\mathcal{G}$(**K**) is called a ***grid-cell*** of **K**.

A convex xel complex **K** *is uniquely determined by* $\mathcal{G}$(**K**)!

Examples of 2D Convex Xel Complexes

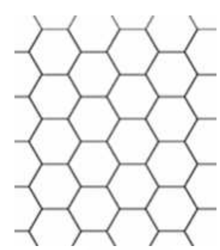
A ***2D cubical xel complex*** is a xel complex **K** for which $\mathcal{G}$(**K**) is a set of *squares* that tessellate the plane in the "obvious" way.



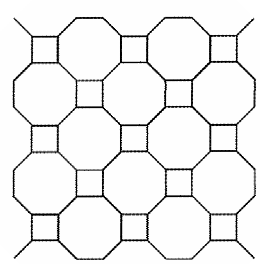
Binary images on the grid cells of this xel complex are just binary images on the familiar "2D Cartesian grid".

If in our main results we take **K** to be a 2D cubical xel complex, we obtain Ronse's two theorems.

A ***2D hexagonal xel complex*** is a xel complex **K** for which $\mathcal{G}$(**K**) is a set of *hexagons* that tessellate the plane like this:

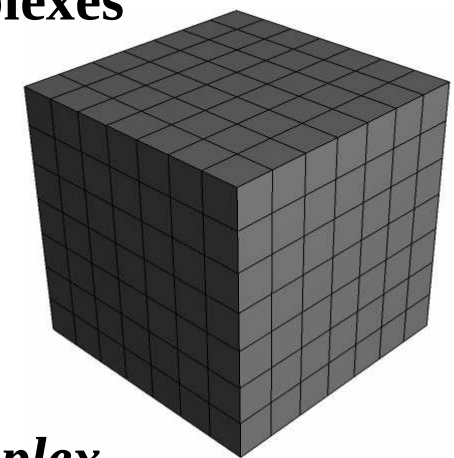


A ***2D Khalimsky xel complex*** is a xel complex **K** for which $\mathcal{G}$(**K**) is a set of *octagons* and *squares* that tessellate the plane as shown on the right.

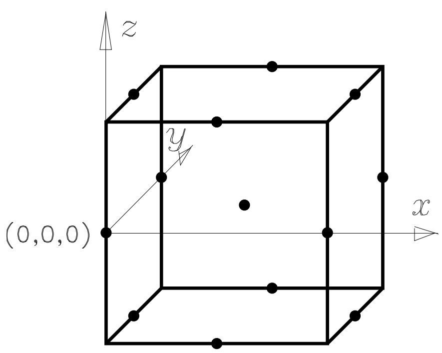


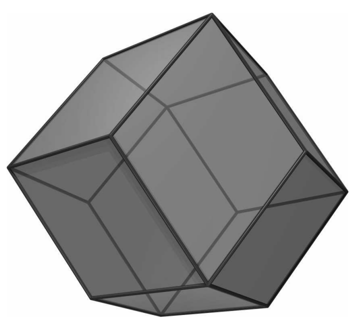
Examples of 3D Convex Xel Complexes

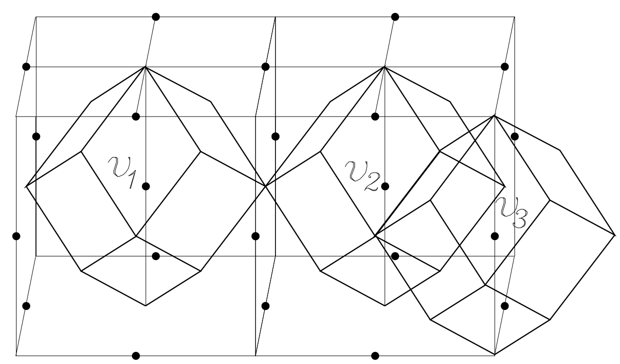
A ***3D cubical xel complex*** is a xel complex **K** for which $\mathcal{G}$(**K**) is a set of *cubes* that tessellate 3-space in the obvious way:



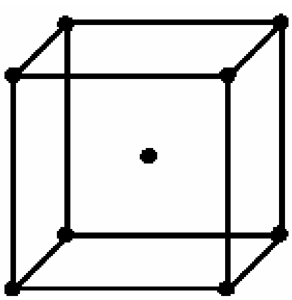
A ***3D face-centered cubical xel complex*** is a xel complex **K** for which $\mathcal{G}$(**K**) is a set of *rhombic dodecahedra* that tessellate 3-space as the Voronoi neighborhoods of a face-centered cubic lattice.

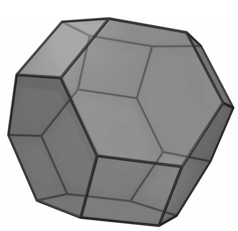


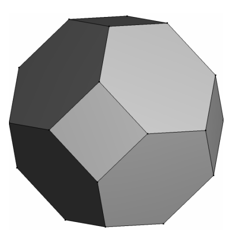


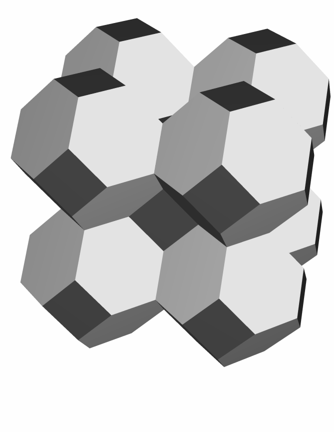


A ***3D body-centered cubical xel complex*** is a xel complex **K** for which $\mathcal{G}$(**K**) is a set of *truncated octahedra* that tessellate 3-space as the Voronoi neighborhoods of a body-centered cubic lattice.









Weak and Strong Components

We now generalize the familiar concepts of "8-" and "4-"adjacency, connectedness, and components to the grid-cells of any xel complex.

If P and Q are grid-cells of an nD xel complex,
we say P is ***weakly adjacent*** to Q if $P \neq Q$ and $P \cap Q \neq \emptyset$;
we say P is ***strongly adjacent*** to Q if $P \cap Q$ is an (n–1)-xel.

Weakly adjacent grid-cells share at least a vertex.
Strongly adjacent grid-cells share an (n–1)D face.

A set $\mathcal{T}$ of grid-cells of a xel complex is said to be ***weakly (strongly) connected*** if, for all $P, Q \in \mathcal{T}$, there exist $T_0, T_1, \ldots, T_m \in \mathcal{T}$ such that $T_0 = P$, $T_m = Q$, and T_i is weakly (strongly) adjacent to T_{i+1} for $0 \leq i < m$.

If $\mathcal{S} \neq \emptyset$ is a set of grid-cells of a xel complex, then each maximal weakly (strongly) connected subset of $\mathcal{S}$ is called a ***weak (strong) component*** of $\mathcal{S}$.

Next, we will generalize the concept of an ***8-deletable set*** to binary images on arbitrary xel complexes ...

Binary Images; Deletable Sets of 1's

A ***binary image*** on a xel complex **K** is a mapping $\mathbb{I} : \mathcal{G}(\mathbf{K}) \to \{0,1\}$ for which either $\mathbb{I}^{-1}[\{0\}]$ or $\mathbb{I}^{-1}[\{1\}]$ is a finite set. [We may omit "binary", for brevity!]

If $P \in \mathcal{G}(\mathbf{K})$ and $\mathbb{I}(P) = 1$, then we say P is a **1** of $\mathbb{I}$.
If $P \in \mathcal{G}(\mathbf{K})$ and $\mathbb{I}(P) = 0$, then we say P is a **0** of $\mathbb{I}$.

Observation: $\mathbb{I}^{-1}[\{1\}]$ = the set of all the 1's of $\mathbb{I}$.

Let $\mathcal{D} \subseteq \mathbb{I}^{-1}[\{1\}]$. Then we say $\mathcal{D}$ is a ***deletable*** set of $\mathbb{I}$ if $\mathcal{D}$ is finite and the *polyhedron* $\bigcup \mathbb{I}^{-1}[\{1\}]$ *can be continuously deformed** *over itself onto the polyhedron* $\bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \mathcal{D})$.

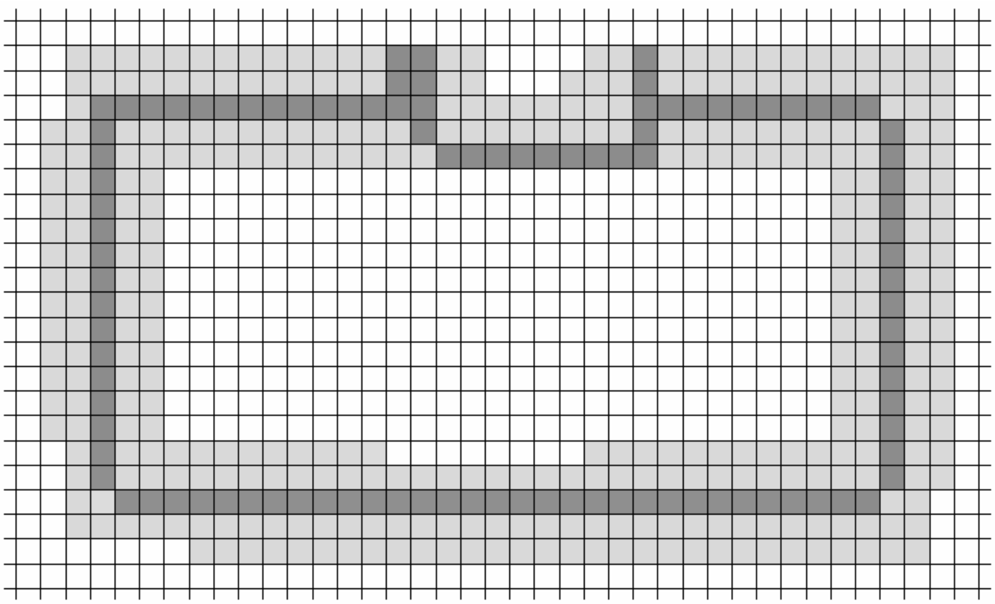
* in such a way that all points in $\bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \mathcal{D})$ remain fixed throughout the deformation process.

In topology, such a continuous deformation process is called a ***deformation retraction***. Hence:

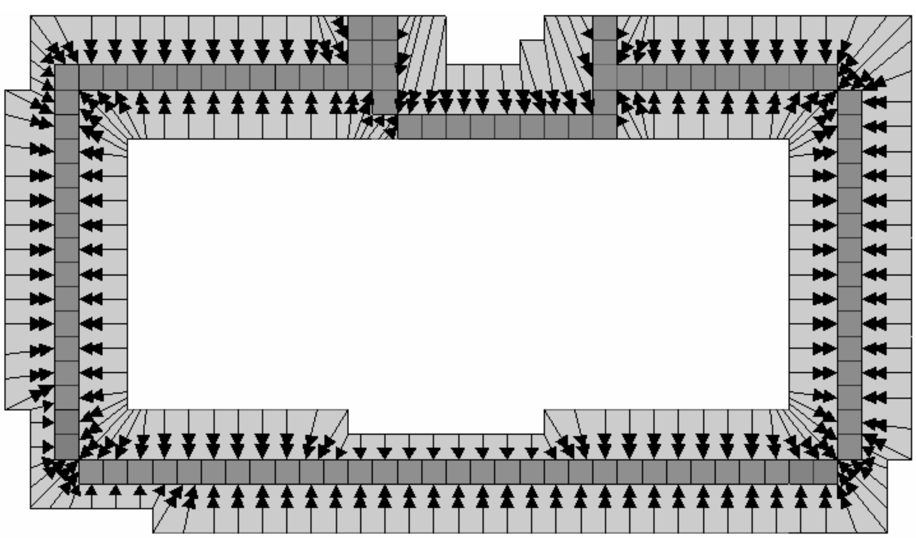
Definition *A set* $\mathcal{D} \subseteq \mathbb{I}^{-1}[\{1\}]$ *is a* **deletable** *set of* $\mathbb{I}$ *if* $\mathcal{D}$ *is finite and there is a deformation retraction of* $\bigcup \mathbb{I}^{-1}[\{1\}]$ *onto* $\bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \mathcal{D})$.

Example of an image $\mathbb{I}$ (on a 2D cubical xel complex) and a ***deletable*** set $\mathcal{D}$ of 1's of $\mathbb{I}$:

- ▢ = a 1 of $\mathbb{I}$ in $\mathcal{D}$
- ■ = a 1 of $\mathbb{I}$ in $\mathbb{I}^{-1}[\{1\}] \setminus \mathcal{D}$
- □ = a 0 of $\mathbb{I}$



$\mathcal{D}$ is a deletable set of 1's of $\mathbb{I}$



A deformation retraction of $\bigcup \mathbb{I}^{-1}[\{1\}]$ onto $\bigcup (\mathbb{I}^{-1}[\{1\}] \setminus \mathcal{D})$

Notation: If $\mathcal{D}$ is a set of 1's of an image $\mathbb{I}$, then

$\mathbb{I} - \mathcal{D} \stackrel{\text{def}}{=}$ the image obtained from $\mathbb{I}$ by changing all the 1's in $\mathcal{D}$ to 0's.

If $\mathcal{A}$ is a set of 0's of an image $\mathbb{I}$, then

$\mathbb{I} + \mathcal{A} \stackrel{\text{def}}{=}$ the image obtained from $\mathbb{I}$ by changing all the 0's in $\mathcal{A}$ to 1's.

Two Properties of Deletable Sets

Changing all the 1'*s in a* *deletable* *set to* 0'*s*

- *preserves weak components of the* 1'*s*
- *preserves strong components of the* 0'*s*

More precisely, if $\mathcal{D}$ is any *deletable* set of 1's of an image $\mathbb{I}$, then:

- *Each weak component of the* 1'*s of* $\mathbb{I}$ *contains just one weak component of the* 1'*s of* $\mathbb{I} - \mathcal{D}$.
- *Each strong component of the* 0'*s of* $\mathbb{I} - \mathcal{D}$ *contains just one strong component of the* 0'*s of* $\mathbb{I}$.

Note: On a 2D cubical xel complex, the weak-(strong-)components are the 8-(4-)components, so the above-mentioned properties imply that the *deletable* sets are exactly the *8-deletable* sets.

Codeletable Sets of 1's

The concept of a ***codeletable set*** generalizes the concept of a 4-*deletable set* to arbitrary xel complexes.

Notation: If $\mathbb{I} : \mathcal{G}(\mathbf{K}) \to \{0,1\}$ is an image, then the image $\mathbb{I}^c : \mathcal{G}(\mathbf{K}) \to \{0,1\}$ is defined by $\mathbb{I}^c(P) = 1 - \mathbb{I}(P)$ for all $P \in \mathcal{G}(\mathbf{K})$.

Note: The 1's and 0's of $\mathbb{I}^c$ are respectively the 0's and 1's of $\mathbb{I}$.

Definition *A set* $\mathcal{S}$ *of* 1'*s of an image* $\mathbb{I}$ *is called a* ***codeletable*** *set of* $\mathbb{I}$ if $\mathcal{S}$ *is a deletable set of* $(\mathbb{I} - \mathcal{S})^c$.

Note: The 1's of the image $(\mathbb{I} - \mathcal{S})^c$ are just the 0's of $\mathbb{I}$ and the elements of the set $\mathcal{S}$; equivalently, $(\mathbb{I} - \mathcal{S})^c - \mathcal{S} = \mathbb{I}^c$.

If $\mathcal{S}$ is any *codeletable* set of 1's of an image $\mathbb{I}$, then changing the elements of $\mathcal{S}$ to 0's "*preserves strong components of the* 1'*s and weak components of the* 0's":

- *Each strong component of the* 1'*s of* $\mathbb{I}$ *contains just one strong component of the* 1'*s of* $\mathbb{I} - \mathcal{S}$.
- *Each weak component of the* 0'*s of* $\mathbb{I} - \mathcal{S}$ *contains just one weak component of the* 0'*s of* $\mathbb{I}$.

Minimal Non-Deletable Sets and Minimal Non-Codeletable Sets of 1's

A ***minimal non-deletable*** set of an image $\mathbb{I}$ is a set $\mathcal{D}$ of 1's of $\mathbb{I}$ such that:

1. Every proper subset of $\mathcal{D}$ is a *deletable* set of $\mathbb{I}$.
2. $\mathcal{D}$ is a *non-deletable* set of $\mathbb{I}$.

A ***minimal non-codeletable*** set of an image $\mathbb{I}$ is a set $\mathcal{D}$ of 1's of $\mathbb{I}$ such that:

1. Every proper subset of $\mathcal{D}$ is a *codeletable* set of $\mathbb{I}$.
2. $\mathcal{D}$ is a *non-codeletable* set of $\mathbb{I}$.

If **K** is a xel complex, then we say that a given set $\mathcal{D}$ of grid-cells of **K** ***can be minimal non-(co)deletable on K*** if there exists an image $\mathbb{I} : \mathcal{G}(\mathbf{K}) \rightarrow \{0,1\}$ such that $\mathcal{D}$ is a minimal non-(co)deletable set of $\mathbb{I}$.

We say that a given set $\mathcal{D}$ of grid-cells of **K** ***can be minimal non-(co)deletable on K as a proper subset of a component*** if there exists an image $\mathbb{I} : \mathcal{G}(\mathbf{K}) \rightarrow \{0,1\}$ such that $\mathcal{D}$ is a minimal non-(co)deletable set of $\mathbb{I}$, and $\mathcal{D}$ is a *proper* subset of a weak (strong) component of the 1's of $\mathbb{I}$.

Our Main Results:

Theorem *Let* **K** *be an nD xel complex* (*where* $n \leq 4$) *and let* $\emptyset \neq \mathcal{D} \subseteq \mathcal{G}(\mathbf{K})$. *Then*:

A1. $\mathcal{D}$ *can be minimal non-deletable on* **K** ***iff*** $\bigcap \mathcal{D} \neq \emptyset$.

A2. $\mathcal{D}$ *can be minimal non-deletable on* **K** *as a proper subset of a component* ***iff*** $\dim(\bigcap \mathcal{D}) \geq 1$.

B1. $\mathcal{D}$ *can be minimal non-codeletable on* **K** ***iff*** $\bigcap \mathcal{D} \neq \emptyset$ *and* $\nexists \mathcal{D}'\ (\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap \mathcal{D}' = \bigcap \mathcal{D})$.

B2. $\mathcal{D}$ *can be minimal non-codeletable on* **K** *as a proper subset of a component* ***iff*** $|\mathcal{D}| \leq n$, $\bigcap \mathcal{D} \neq \emptyset$, *and* $\nexists \mathcal{D}'\ (\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap \mathcal{D}' = \bigcap \mathcal{D})$.

Note: A2 $\Rightarrow$ A1 B2 $\Rightarrow$ B1

Trivial Examples:

(a) If $\mathcal{D}$ is a singleton, then $\mathcal{D}$ satisfies A2 and B2.
[For A2, this is because $\bigcap \mathcal{D} = \mathcal{D}$.]

(b) If there exist $P, Q \in \mathcal{D}$ such that P and Q are *not* weakly adjacent, then $\mathcal{D}$ satisfies *none* of the four conditions.
[In this case $\bigcap \mathcal{D} = \emptyset$.]

Theorem *Let* **K** *be an nD xel complex* (*where* $n \leq 4$)
and let $\emptyset \neq \mathcal{D} \subseteq \mathcal{G}(\mathbf{K})$. *Then*:

A1. $\mathcal{D}$ *can be minimal non-deletable on* **K** ***iff*** $\bigcap \mathcal{D} \neq \emptyset$.

A2. $\mathcal{D}$ *can be minimal non-deletable on* **K** *as a proper subset of a component* ***iff*** $\dim(\bigcap \mathcal{D}) \geq 1$.

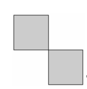
B1. $\mathcal{D}$ *can be minimal non-codeletable on* **K**
iff $\bigcap \mathcal{D} \neq \emptyset$ *and* $\nexists \mathcal{D}' \; (\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap \mathcal{D}' = \bigcap \mathcal{D})$.

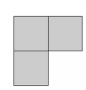
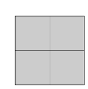
B2. $\mathcal{D}$ *can be minimal non-codeletable on* **K** *as a proper subset of a component*
iff $|\mathcal{D}| \leq n$, $\bigcap \mathcal{D} \neq \emptyset$, *and* $\nexists \mathcal{D}' \; (\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap \mathcal{D}' = \bigcap \mathcal{D})$.

More Examples

(c) If $n \geq 2$ and $\mathcal{D}$ is a pair of <u>*strongly adjacent*</u> grid-cells $\{P, Q\}$, then $\mathcal{D}$ satisfies A2 and B2.

[For A2, this is because $\bigcap \mathcal{D} = P \cap Q$ is an (n–1)-xel;
for B2, note that in this case
$$\emptyset \neq \mathcal{D}' \subsetneq \mathcal{D} \;\Rightarrow\; \mathcal{D}' = \{P\} \text{ or } \{Q\} \;\Rightarrow\; \bigcap \mathcal{D}' \neq \bigcap \mathcal{D}.]$$

(d) If **K** is a 2D cubical xel complex, and $\mathcal{D} =$ ,
then $\mathcal{D}$ satisfies A1 but not A2,
and $\mathcal{D}$ satisfies B2.

(e) If **K** is a 2D cubical xel complex, and $\mathcal{D} =$  or ,
then $\mathcal{D}$ satisfies A1 but not A2,
and $\mathcal{D}$ does not satisfy the B conditions.

Theorem *Let* **K** *be an nD xel complex* (*where* $n \leq 4$) *and let* $\emptyset \neq \mathcal{D} \subseteq \mathcal{G}(\mathbf{K})$. *Then*:

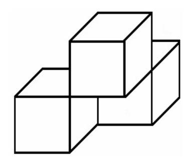
A1. $\mathcal{D}$ *can be minimal non-deletable on* **K** ***iff*** $\bigcap\mathcal{D} \neq \emptyset$.

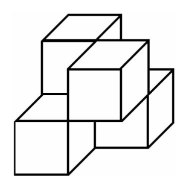
A2. $\mathcal{D}$ *can be minimal non-deletable on* **K** *as a proper subset of a component* ***iff*** $\dim(\bigcap\mathcal{D}) \geq 1$.

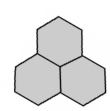
B1. $\mathcal{D}$ *can be minimal non-codeletable on* **K** ***iff*** $\bigcap\mathcal{D} \neq \emptyset$ *and* $\nexists\mathcal{D}'$ $(\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap\mathcal{D}' = \bigcap\mathcal{D})$.

B2. $\mathcal{D}$ *can be minimal non-codeletable on* **K** *as a proper subset of a component* ***iff*** $|\mathcal{D}| \leq n$, $\bigcap\mathcal{D} \neq \emptyset$, *and* $\nexists\mathcal{D}'$ $(\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap\mathcal{D}' = \bigcap\mathcal{D})$.

Further Examples

(f) If **K** is a 3D cubical xel complex, and $\mathcal{D}$ = 
then $\mathcal{D}$ satisfies A1 but not A2,
and $\mathcal{D}$ satisfies B2.

(g) If **K** is a 3D cubical xel complex, and $\mathcal{D}$ = 
then $\mathcal{D}$ satisfies A1 but not A2,
and $\mathcal{D}$ does not satisfy the B conditions.

(h) If **K** is a 2D hexagonal xel complex, and $\mathcal{D}$ = 
then $\mathcal{D}$ satisfies A1 and B1,
but $\mathcal{D}$ does not satisfy A2 or B2.

Theorem *Let* **K** *be an nD xel complex* (*where* $n \le 4$) *and let* $\emptyset \ne \mathcal{D} \subseteq \mathcal{G}(\mathbf{K})$. *Then*:

A1. $\mathcal{D}$ *can be minimal non-deletable on* **K** ***iff*** $\bigcap\mathcal{D} \ne \emptyset$.

A2. $\mathcal{D}$ *can be minimal non-deletable on* **K** *as a proper subset of a component* ***iff*** $\dim(\bigcap\mathcal{D}) \ge 1$.

B1. $\mathcal{D}$ *can be minimal non-codeletable on* **K** ***iff*** $\bigcap\mathcal{D} \ne \emptyset$ *and* $\nexists \mathcal{D}'$ $(\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap\mathcal{D}' = \bigcap\mathcal{D})$.

B2. $\mathcal{D}$ *can be minimal non-codeletable on* **K** *as a proper subset of a component* ***iff*** $|\mathcal{D}| \le n$, $\bigcap\mathcal{D} \ne \emptyset$, *and* $\nexists \mathcal{D}'$ $(\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap\mathcal{D}' = \bigcap\mathcal{D})$.

Lemma: *The condition of* B1 *implies* $|\mathcal{D}| \le n+1$.

Proof: Let $k = |\mathcal{D}| - 1$, let $\mathcal{D} = \{P_0, P_1, ..., P_k\}$, and suppose the condition of B1 holds. We must show $k \le n$.

No two members of the descending chain

$$P_0 \supseteq P_0 \cap P_1 \supseteq P_0 \cap P_1 \cap P_2 \supseteq \ ... \supseteq P_0 \cap P_1 \cap \ ... \cap P_k = \bigcap\mathcal{D} \ne \emptyset$$

can be equal — for if $P_0 \cap \ ... \cap P_i = P_0 \cap \ ... \cap P_i \cap P_{i+1}$ then

$$\begin{aligned}\bigcap\mathcal{D} &= (P_0 \cap \ ... \cap P_{i+1}) \cap (P_{i+2} \cap \ ... \cap P_k)\\ &= (P_0 \cap \ ... \cap P_i) \cap (P_{i+2} \cap \ ... \cap P_k)\\ &= \bigcap(\mathcal{D} \setminus \{P_{i+1}\}),\text{ contrary to the condition of B1.}\end{aligned}$$

So, since each intersection $P_0 \cap \ ... \cap P_i$ is a xel, we have $\dim(P_0 \cap \ ... \cap P_{i+1}) \le \dim(P_0 \cap \ ... \cap P_i) - 1$ for all $i < k$ whence $\dim(\bigcap\mathcal{D}) = \dim(P_0 \cap \ ... \cap P_k) \le \dim(P_0) - k = n-k$. Therefore $n-k \ge \dim(\bigcap\mathcal{D}) \ge 0$. //

Theorem *Let* **K** *be an nD xel complex* (*where* $n \le 4$) *and let* $\emptyset \neq \mathcal{D} \subseteq \mathcal{G}(\mathbf{K})$. *Then*:

A1. $\mathcal{D}$ *can be minimal non-deletable on* **K** ***iff*** $\bigcap\mathcal{D} \neq \emptyset$.

A2. $\mathcal{D}$ *can be minimal non-deletable on* **K** *as a proper subset of a component* ***iff*** $\dim(\bigcap\mathcal{D}) \geq 1$.

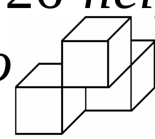
B1. $\mathcal{D}$ *can be minimal non-codeletable on* **K** ***iff*** $\bigcap\mathcal{D} \neq \emptyset$ *and* $\nexists\mathcal{D}'$ $(\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap\mathcal{D}' = \bigcap\mathcal{D})$.

B2. $\mathcal{D}$ *can be minimal non-codeletable on* **K** *as a proper subset of a component* ***iff*** $|\mathcal{D}| \le n$, $\bigcap\mathcal{D} \neq \emptyset$, *and* $\nexists\mathcal{D}'$ $(\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap\mathcal{D}' = \bigcap\mathcal{D})$.

When **K** is a ***3D cubical xel complex***:

$\mathcal{D}$ satisfies A1 ***iff*** $\emptyset \neq \mathcal{D} \subseteq$ *a* 2×2×2 *block of* 8 *voxels*.

$\mathcal{D}$ satisfies A2 ***iff*** $\emptyset \neq \mathcal{D} \subseteq$ *a* 2×2×1 *block of* 4 *voxels*.

$\mathcal{D}$ satisfies B2 ***iff*** $\mathcal{D}$ *is a singleton*, or
$\mathcal{D}$ *is a pair of* 26-*neighbors*, or
$\mathcal{D}$ *isometric to* .

Note: *On a* <u>*cubical*</u> *xel complex* (2*D*, 3*D*, *or* 4*D*), *any set* $\mathcal{D}$ *that satisfies* B1 *also satisfies* B2.

$\mathcal{D}$ satisfies B1 but not B2 ***iff***
$|\mathcal{D}| = n+1$ (where $n = \dim(\mathbf{K})$) and, for $1 \le k \le n$, the $\bigcap$ of any $k+1$ members of $\mathcal{D}$ is an $(n-k)$-xel.

In an *n*D *cubical* xel complex with $n \geq 2$, <u>*no such sets exist*</u> because $P_0 \cap P_1$, $P_0 \cap P_2$, and $P_1 \cap P_2$ cannot all be $(n-1)$-xels if the *P*'s are distinct grid-cells.

Simple 1's and *Cosimple* 1's in Binary Images

The concept of a *simple* 1 generalizes the concept of an *8-simple* 1 to images on arbitrary xel complexes:

Let P be a 1 of an image $\mathbb{I}$.

We say P is a ***simple*** 1 of $\mathbb{I}$ if the singleton set $\{P\}$ is a deletable set of $\mathbb{I}$.

Observation:

P is a simple 1 of $\mathbb{I}$
$\Leftrightarrow$
There is a deformation retraction of $\bigcup \mathbb{I}^{-1}[\{1\}]$ onto $\bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \{P\})$.

The concept of a *co*simple 1 generalizes the concept of a *4-simple* 1 to images on arbitrary xel complexes:

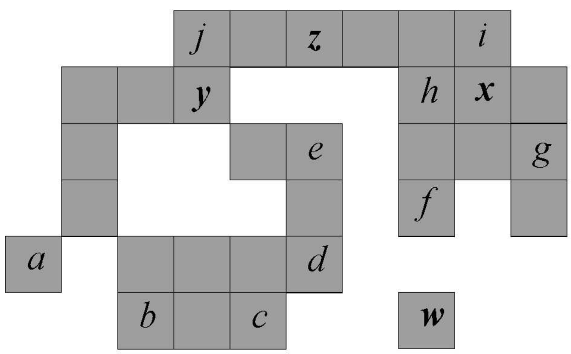
We say P is a ***cosimple*** 1 of an image $\mathbb{I}$ if P is a simple 1 of the image $(\mathbb{I} - \{P\})^{c}$.

Hence P is a cosimple 1 of an image $\mathbb{I}$ ***iff*** the singleton set $\{P\}$ is a codeletable set of $\mathbb{I}$.

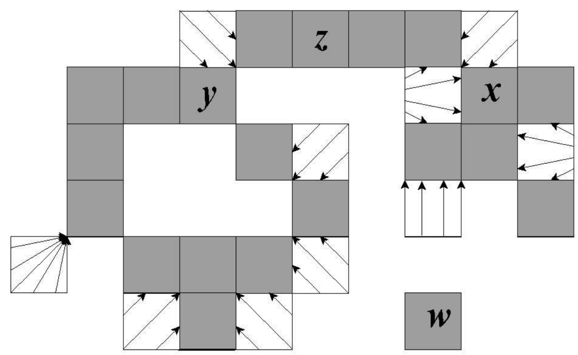
Remark: In an image on a 3D cubical xel complex, the concepts of *simple* and *cosimple* 1's are equivalent to the standard concepts of *26-simple* and *6-simple* 1's.

2D Illustrations of:

P is a simple 1 of $\mathbb{I}$ $\Leftrightarrow$ There is a deformation retraction of $\bigcup \mathbb{I}^{-1}[\{1\}]$ onto $\bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \{P\})$.



$a, b, \ldots, j$ are some ***simple*** 1's of an image $\mathbb{I}$ (whose 1's are the gray squares); w, x, y, z are some ***non-simple*** 1's of $\mathbb{I}$.



Deformation retractions of $\bigcup\mathbb{I}^{-1}[\{1\}]$ onto $\bigcup(\mathbb{I} \setminus\{P\})$, for $P \in \{a, b, ..., j\}$.

Simple Sets and *Cosimple* Sets of 1's

A set $\mathcal{P}$ of 1's of an image $\mathbb{I}$ is said to be a ***simple set*** of $\mathbb{I}$ if the elements of $\mathcal{P}$ can be arranged in a sequence $Q_1, Q_2, ..., Q_k$ in which Q_1 is a simple 1 of $\mathbb{I}$ and each Q_i $(i > 1)$ is a simple 1 of $\mathbb{I} - \{Q_1, Q_2, ..., Q_{i-1}\}$.

Here is an equivalent recursive definition:

Definition ***Simple sets*** *of* 1*'s in an image* $\mathbb{I}$ *are defined recursively, as follows*:

- $\varnothing$ *is a simple set of* $\mathbb{I}$.
- *If* $\mathcal{D}$ *is a set of* 1*'s of* $\mathbb{I}$, *and there is some* $Q \in \mathcal{D}$ *for which*
    1. $\mathcal{D} \setminus \{Q\}$ *is a simple set of* $\mathbb{I}$, and
    2. Q *is a simple* 1 *of* $\mathbb{I} - (\mathcal{D} \setminus \{Q\})$

  then $\mathcal{D}$ *is a simple set of* $\mathbb{I}$.

Cosimple sets of 1's are defined analogously—just replace "simple" with "cosimple" in the above.

$\mathcal{D}$ is a (co)simple set of $\mathbb{I}$ $\Rightarrow$ $\mathcal{D}$ is a (co)deletable set of $\mathbb{I}$

Moreover, in any image $\mathbb{I}$ on a **2D** xel complex,

$\mathcal{D}$ is a (co)simple set of $\mathbb{I}$ $\Leftrightarrow$ $\mathcal{D}$ is a (co)deletable set of $\mathbb{I}$

Minimal Non-Deletable ≡ Minimal Non-Simple

A ***minimal non-(co)simple*** set of an image $\mathbb{I}$ is a set $\mathcal{D}$ of 1's of $\mathbb{I}$ such that:

1. Each proper subset of $\mathcal{D}$ is a (co)simple set of $\mathbb{I}$.
2. $\mathcal{D}$ is a non-(co)simple set of 1's of $\mathbb{I}$.

Even though a (co)deletable set need not be a (co)simple set, we can show that:

$\mathcal{D}$ is a *minimal non*-(co)*deletable set* of $\mathbb{I}$ $\Leftrightarrow$ $\mathcal{D}$ is a *minimal non*-(co)*simple set* of $\mathbb{I}$	(★)

For **X** = simple, cosimple, deletable, or codeletable, we say that a set $\mathcal{D}$ of 1's of $\mathbb{I}$ is ***hereditarily* X** if every subset of $\mathcal{D}$ has the property **X**:

$\mathcal{D}$ is hereditarily **X** $\Leftrightarrow$ $\forall \mathcal{D}' \subseteq \mathcal{D}$ $\mathcal{D}'$ has the property **X**

Evidently, a set $\mathcal{D}$ is *minimal non*-**X** if and only if

- $\mathcal{D}$ is ***not*** hereditarily **X**, but
- every proper subset of $\mathcal{D}$ is hereditarily **X**

Hence (★) can be proved by showing:

hereditarily (co)simple $\Leftrightarrow$ hereditarily (co)deletable

Since "*is a simple set*" $\Rightarrow$ "*is a deletable set*"
we have that

"*is hereditarily simple*" $\Rightarrow$ "*is hereditarily deletable*"

To prove the reverse implication

"*is hereditarily deletable*" $\Rightarrow$ "*is hereditarily simple*"

it is enough to prove

"*is hereditarily deletable*" $\Rightarrow$ "*is simple*" (†)

(†) is a consequence of:

Lemma: *Let* $\mathcal{D}$ *be a set of* 1's *of an image* $\mathbb{I}$ and let $Q \in \mathcal{D}$. *Then any two of the following imply the third*:

1. $\mathcal{D} \setminus \{Q\}$ is a (co)deletable set of $\mathbb{I}$.
2. Q is a (co)simple 1 of $\mathbb{I} - (\mathcal{D} \setminus \{Q\})$.
3. $\mathcal{D}$ is a (co)deletable set of $\mathbb{I}$.

To deduce (†), let the set $\mathcal{D}$ be a *minimal* counterexample to (†) in an image $\mathbb{I}$. Then $\mathcal{D}$ is hereditarily deletable in $\mathbb{I}$, but <u>$\mathcal{D}$ is not simple in $\mathbb{I}$</u>. Let Q be any element of $\mathcal{D}$.

As $\mathcal{D}$ is hereditarily deletable, both $\mathcal{D}$ and its subset $\mathcal{D} \setminus \{Q\}$ are deletable in $\mathbb{I}$. So, by the above lemma,

Q is a simple 1 of $\mathbb{I} - (\mathcal{D} \setminus \{Q\})$ (⋆)

As $\mathcal{D}$ is a *minimal* counterexample to (†), and $\mathcal{D} \setminus \{Q\}$ is deletable, (†) implies $\mathcal{D} \setminus \{Q\}$ is a simple set of $\mathbb{I}$; this and (⋆) imply <u>$\mathcal{D}$ is simple in $\mathbb{I}$</u>, which is a contradiction.//

The equivalence

hereditarily ***co***deletable $\Leftrightarrow$ hereditarily ***co***simple

can be proved in a similar way, using the same lemma.

Lemma: *Let* $\mathcal{D}$ *be a set of* 1's *of an image* $\mathbb{I}$ and let $Q \in \mathcal{D}$.
Then any two of the following imply the third:

1. $\mathcal{D}\setminus\{Q\}$ is a (co)deletable set of $\mathbb{I}$.
2. Q is a (co)simple 1 of $\mathbb{I} - (\mathcal{D}\setminus\{Q\})$.
3. $\mathcal{D}$ is a (co)deletable set of $\mathbb{I}$.

This fundamental lemma is a consequence of the following fact about polyhedra:

Fact: If X, Y, and Z are polyhedra such that $Z \subseteq Y \subseteq X$, then *any two of the following imply the third*:

- Y is a deformation retract of X.
- Z is a deformation retract of Y.
- Z is a deformation retract of X.

Note: On a 3D cubical xel complex, hereditarily simple and cosimple sets are cases of Bertrand's ***$\mathcal{P}$-simple*** sets.

[G. Bertrand, *C. R. Acad. Sci. Paris, Série I* **321**, 1995, 1077–84]

If $\mathcal{P}$ is a set of 1's of a binary image $\mathbb{I}$ on a 3D cubical xel complex, then:

$\mathcal{P}$ *is hereditarily simple in* $\mathbb{I}$ ***iff*** $\mathcal{P}$ *is* $\mathcal{P}_{26}$*-simple in* $\mathbb{I}$.

$\mathcal{P}$ *is hereditarily cosimple in* $\mathbb{I}$ ***iff*** $\mathcal{P}$ *is* $\mathcal{P}_{6}$*-simple in* $\mathbb{I}$.

The Attachment Set of a 1 in a Binary Image $\mathbb{I}$

Let Q be a 1 of an image $\mathbb{I} : \mathcal{G}(\mathbf{K}) \to \{0, 1\}$.

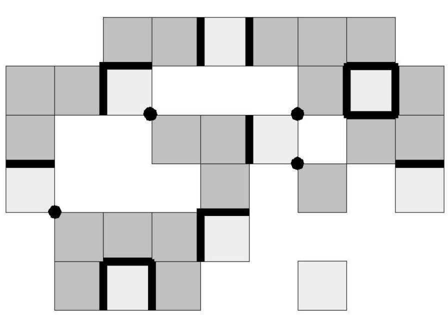
We define two sets of boundary faces of Q:

$$\mathbf{Attach}(Q, \mathbb{I}) \stackrel{\text{def}}{=} \bigcup\{\mathbf{bdryfaces}(Q) \cap \mathbf{bdryfaces}(X) \mid X \in \mathbb{I}^{-1}[\{1\}] \setminus \{Q\}\}$$

$$\mathbf{Coattach}(Q, \mathbb{I}) \stackrel{\text{def}}{=} \bigcup\{\mathbf{bdryfaces}(Q) \cap \mathbf{bdryfaces}(X) \mid X \in \mathbb{I}^{-1}[\{0\}]\ \}$$

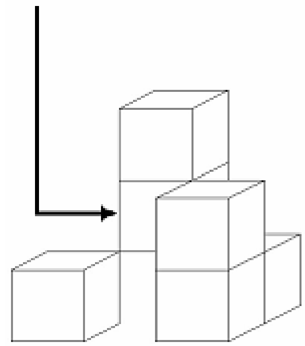
The sets $\bigcup$**Attach**(Q, $\mathbb{I}$) and $\bigcup$**Coattach**(Q, $\mathbb{I}$) will respectively be called the ***attachment set*** of Q in $\mathbb{I}$ and the ***coattachment set*** of Q in $\mathbb{I}$.

If $\bigcup \mathbb{I}^{-1}[\{1\}]$ is obtained by "gluing" Q onto $\bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \{Q\})$, then *the attachment set* $\bigcup$**Attach**(Q, $\mathbb{I}$) *is the set of points at which glue may (usefully) be applied*!

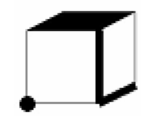


This diagram shows the ***attachment set of each light gray* 1** in the image (on a 2D cubical xel complex) whose 1's are the light gray and dark gray 2-xels.

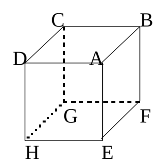
If $\mathbb{I}$ is the image (on a 3D cubical xel complex) whose 1's are shown below, and Q is ***this*** 1

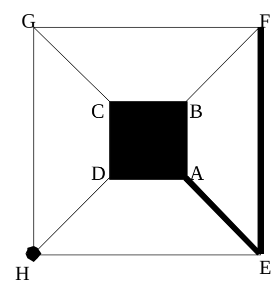


. . . then $\bigcup$**Attach**$(Q, \mathbb{I})$ is this set:



And $\bigcup$**Attach**$(Q, \mathbb{I})$ can be represented by a 2D Schlegel diagram as follows:





If X is any 1 of an image $\mathbb{I}$, then it is straightforward to verify that:

- **Attach**$(X, \mathbb{I}) = \varnothing$
  $\Leftrightarrow$ no other 1 of $\mathbb{I}$ is weakly adjacent to X
- $\bigcup$**Attach**$(X, \mathbb{I}) \quad = \; X \cap \bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \{X\})$
- $\bigcup$**Coattach**$(X, \mathbb{I}) \; = \; X \cap \bigcup(\mathbb{I}^{-1}[\{0\}])$

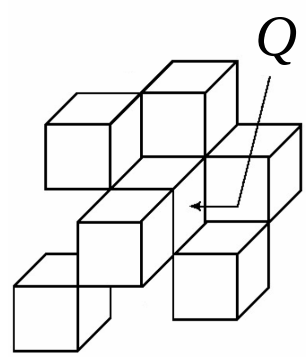
A Local (and Essentially Discrete) Characterization of Simple 1's and Cosimple 1's

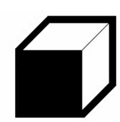
Theorem 1 *If Q is a* 1 *of an image* $\mathbb{I}$ *on an nD xel complex, where $n \leq 4$, then*:
(a) *Q is a simple* 1 ***iff*** $\bigcup$**Attach**$(Q, \mathbb{I})$ *is contractible*.
(b) *Q is a cosimple* 1 ***iff*** $\bigcup$**Coattach**$(Q, \mathbb{I})$ *is contractible*.

In the 4D case, $\bigcup$**Attach**$(Q, \mathbb{I})$ and $\bigcup$**Coattach**$(Q, \mathbb{I})$ can be visualized as subsets of $\mathbb{R}^3 \cup \{\infty\}$ in a Schlegel diagram of **bdryfaces**(Q)!

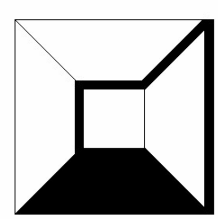
Theorem 1 follows from results of algebraic topology.
[The "if" parts are true even without the hypothesis that $n \leq 4$, but the "only if" parts are not!]

3D Example: Let $\mathbb{I}$ be the image, on a 3D cubical xel complex, whose 1's are shown here. (All hidden voxels are 0's of $\mathbb{I}$.)





$\bigcup$**Attach**$(Q, \mathbb{I})$



Then the attachment set $\bigcup$**Attach**$(Q, \mathbb{I})$ **is *not contractible***, (because it is not simply connected — it "has a hole").
So, by the above theorem, Q is a ***non-simple*** 1 of $\mathbb{I}$.

Characterizations of Minimal Non-Simple (MNS) and Minimal Non-Cosimple (MNCS) Sets

The following theorem states useful necessary and sufficient conditions for a set of 1's to be an MNS or MNCS set:

Theorem 2 *In any image* $\mathbb{I}$ *on a xel complex*:

1. $\mathcal{D}$ *is an MNS set of* $\mathbb{I}$ *if and only if* $\mathcal{D}$ *is finite but nonempty and,* <u>*for every* $Q \in \mathcal{D}$</u>:
   MNS1: Q *is a non-simple* 1 *of* $\mathbb{I} - (\mathcal{D} \setminus \{Q\})$.
   MNS2: Q *is a simple* 1 *of* $\mathbb{I} - \mathcal{D}'$ *whenever* $\mathcal{D}' \subsetneq \mathcal{D} \setminus \{Q\}$.

2. $\mathcal{D}$ *is an MNCS set of* $\mathbb{I}$ *if and only if* $\mathcal{D}$ *is finite but nonempty and,* <u>*for every* $Q \in \mathcal{D}$</u>:
   MNCS1: Q *is a non-cosimple* 1 *of* $\mathbb{I} - (\mathcal{D} \setminus \{Q\})$.
   MNCS2: Q *is a cosimple* 1 *of* $\mathbb{I} - \mathcal{D}'$ *whenever* $\mathcal{D}' \subsetneq \mathcal{D} \setminus \{Q\}$.

Theorem 2 follows from an earlier lemma:

Lemma: *Let* $\mathcal{D}$ *be a set of* 1's *of an image* $\mathbb{I}$ and let $Q \in \mathcal{D}$. *Then any two of the following imply the third*:

1. $\mathcal{D} \setminus \{Q\}$ is a (co)deletable set of $\mathbb{I}$.
2. Q is a (co)simple 1 of $\mathbb{I} - (\mathcal{D} \setminus \{Q\})$.
3. $\mathcal{D}$ is a (co)deletable set of $\mathbb{I}$.

Recall: **Thm. 1** *If Q is a 1 of an image $\mathbb{I}$ on an nD xel complex, where $n \le 4$, then*:

(a) *Q is a simple* 1 ***iff*** $\bigcup$**Attach**$(Q, \mathbb{I})$ *is contractible*.

(b) *Q is a cosimple* 1 ***iff*** $\bigcup$**Coattach**$(Q, \mathbb{I})$ *is contractible*.

Thm. 2 *In any image $\mathbb{I}$ on a xel complex*:

1. *$\mathcal{D}$ is an MNS set of $\mathbb{I}$ if and only if $\mathcal{D}$ is finite but nonempty and, for every $Q \in \mathcal{D}$*:
   MNS1: *Q is a non-simple* 1 *of* $\mathbb{I} - (\mathcal{D} \setminus \{Q\})$.
   MNS2: *Q is a simple* 1 *of* $\mathbb{I} - \mathcal{D}'$ *whenever* $\mathcal{D}' \subsetneq \mathcal{D} \setminus \{Q\}$.
2. *$\mathcal{D}$ is an MNCS set of $\mathbb{I}$ if and only if $\mathcal{D}$ is finite but nonempty and, for every $Q \in \mathcal{D}$*:
   MNCS1: *Q is a non-cosimple* 1 *of* $\mathbb{I} - (\mathcal{D} \setminus \{Q\})$.
   MNCS2: *Q is a cosimple* 1 *of* $\mathbb{I} - \mathcal{D}'$ *whenever* $\mathcal{D}' \subsetneq \mathcal{D} \setminus \{Q\}$.

Combining Theorems 1 and 2, we obtain:

Theorem *Let $\mathcal{D}$ be a set of* 1*'s of an image $\mathbb{I}$ on an nD xel complex, where $n \le 4$. Then*:

1. *$\mathcal{D}$ is an MNS set of $\mathbb{I}$ if and only if $\mathcal{D}$ is finite but nonempty and, for every $Q \in \mathcal{D}$*:
   (a) $\bigcup$**Attach**$(Q, \mathbb{I} - (\mathcal{D} \setminus \{Q\}))$ *is **not** contractible*.
   (b) $\bigcup$**Attach**$(Q, \mathbb{I} - \mathcal{D}')$ *is contractible whenever* $\mathcal{D}' \subsetneq \mathcal{D} \setminus \{Q\}$.

2. *$\mathcal{D}$ is an MNCS set of $\mathbb{I}$ if and only if $\mathcal{D}$ is finite but nonempty and, for every $Q \in \mathcal{D}$*:
   (a) $\bigcup$**Coattach**$(Q, \mathbb{I} - (\mathcal{D} \setminus \{Q\}))$ *is **not** contractible*.
   (b) $\bigcup$**Coattach**$(Q, \mathbb{I} - \mathcal{D}')$ *is contractible whenever* $\mathcal{D}' \subsetneq \mathcal{D} \setminus \{Q\}$.

Recall:

1. *$\mathcal{D}$ is an MNS set of $\mathbb{I}$ iff $\mathcal{D}$ is finite but nonempty and, <u>for every $Q \in \mathcal{D}$</u>:*
   (a) $\bigcup$**Attach**$(Q, \mathbb{I} - (\mathcal{D} \setminus \{Q\}))$ *is **not** contractible.*
   (b) $\bigcup$**Attach**$(Q, \mathbb{I} - \mathcal{D}')$ *is contractible whenever $\mathcal{D}' \subsetneq \mathcal{D} \setminus \{Q\}$.*

Let $\mathbb{I}$ be an image on an nD xel complex, where $n \le 4$.
Let $\mathcal{D}$ be a nonempty finite set of 1's of $\mathbb{I}$, and
let $k = |\mathcal{D}| - 1$.

For each $Q \in \mathcal{D}$, define A_Q, T_Q^i, and X_Q^i as follows:

Let A_Q denote the set $\bigcup$**Attach**$(Q, \mathbb{I} - (\mathcal{D} \setminus \{Q\}))$.

Let $(T_Q^i \,|\, 1 \le i \le k)$ be an enumeration of $\mathcal{D} \setminus \{Q\}$.

Let $X_Q^i = Q \cap T_Q^i$ (for $1 \le i \le k$).

Then, for any subset $\{i_1, ..., i_r\}$ *of* $\{1, ..., k\}$,

$A_Q \cup X_Q^{i_1} \cup ... \cup X_Q^{i_r} = \bigcup$**Attach**$(Q, \mathbb{I} - \mathcal{D}')$

where $\mathcal{D}' = (\mathcal{D} \setminus \{Q\}) \setminus \{T_Q^{i_1}, ..., T_Q^{i_r}\}$.

From the above, we deduce:

$\mathcal{D}$ is an MNS set of $\mathbb{I}$ ***if and only if***, for all $Q \in \mathcal{D}$:

(1) A_Q is not contractible, but

(2) $A_Q \cup X_Q^{i_1} \cup ... \cup X_Q^{i_r}$ is contractible
for all <u>*nonempty*</u> subsets $\{i_1, ..., i_r\}$ of $\{1, ..., k\}$.

Recall: Let $\mathcal{D}$ be a nonempty finite set of 1's of $\mathbb{I}$.
Let $k = |\mathcal{D}| - 1$ and, for each $Q \in \mathcal{D}$,
let A_Q denote the set $\bigcup$**Attach**$(Q, \mathbb{I} - (\mathcal{D} \setminus \{Q\}))$,
let $(T_Q^i \mid 1 \le i \le k)$ be an enumeration of $\mathcal{D} \setminus \{Q\}$, and
let $X_Q^i = Q \cap T_Q^i$ (for $1 \le i \le k$).

Then $\mathcal{D}$ is an MNS set of $\mathbb{I}$ ***if and only if***, for all $Q \in \mathcal{D}$:

(1) A_Q is not contractible, but
(2) $A_Q \cup X_Q^{i_1} \cup \ldots \cup X_Q^{i_r}$ is contractible
for all <u>*nonempty*</u> subsets $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, k\}$.

Similarly, if C_Q denotes the set $\bigcup$**Coattach**$(Q, \mathbb{I})$, then:
$\mathcal{D}$ is an MNCS set of $\mathbb{I}$ ***if and only if***, for all $Q \in \mathcal{D}$:

(1') $C_Q \cup X_Q^1 \cup \ldots \cup X_Q^k$ is not contractible, but

(2') $C_Q \cup X_Q^{i_1} \cup \ldots \cup X_Q^{i_r}$ is contractible
for all <u>*proper*</u> subsets $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, k\}$.

Note: $C_Q \cup X_Q^{i_1} \cup \ldots \cup X_Q^{i_r} = \bigcup$**Coattach**$(Q, \mathbb{I} - \{T_Q^{i_1}, \ldots, T_Q^{i_r}\})$
$C_Q \cup X_Q^1 \cup \ldots \cup X_Q^k = \bigcup$**Coattach**$(Q, \mathbb{I} - (\mathcal{D} \setminus \{Q\}))$

Recall: Let $\mathcal{D}$ be a nonempty finite set of 1's of $\mathbb{I}$, and let $k = |\mathcal{D}| - 1$. For each $Q \in \mathcal{D}$, let $(T_Q^i \mid 1 \le i \le k)$ be an enumeration of $\mathcal{D} \setminus \{Q\}$, let $X_Q^i = Q \cap T_Q^i$ (for $1 \le i \le k$), let $C_Q = \bigcup\textbf{Coattach}(Q, \mathbb{I})$, and let $A_Q = \bigcup\textbf{Attach}(Q, \mathbb{I} - (\mathcal{D} \setminus \{Q\}))$. Then:

A. *$\mathcal{D}$ is an MNS set of $\mathbb{I}$* ***if and only if****, for all $Q \in \mathcal{D}$,*

(1) *A_Q is not contractible, but*

(2) *$A_Q \cup X_Q^{i_1} \cup \ldots \cup X_Q^{i_r}$ is contractible*

for each <u>nonempty</u> subset $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, k\}$.

B. *$\mathcal{D}$ is an MNCS set of $\mathbb{I}$* ***if and only if****, for all $Q \in \mathcal{D}$,*

(1') *$C_Q \cup X_Q^1 \cup \ldots \cup X_Q^k$ is not contractible, but*

(2') *$C_Q \cup X_Q^{i_1} \cup \ldots \cup X_Q^{i_r}$ is contractible*

for each <u>proper</u> subset $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, k\}$.

Key Fact: *Let P_1 and P_2 be polyhedra in 3-space. Then <u>any two of the following imply the third</u>:*

- *Each of P_1 and P_2 is contractible.*
- *$P_1 \cup P_2$ is contractible.*
- *$P_1 \cap P_2$ is contractible.*

An inductive argument based on the Key Fact yields:

Lemma 1: *For any finite collection $\mathcal{S}$ of polyhedra in 3-space, the following are equivalent*:

(a) *$\bigcap\mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subseteq \mathcal{S}$.*

(b) *$\bigcup\mathcal{T}$ is contractible whenever $\emptyset \neq \mathcal{T} \subseteq \mathcal{S}$.*

This lemma and the facts **A** and **B** above are the principal ingredients of our proof of the main results.

Recall: For each $Q \in \mathcal{D}$: A_Q denotes the set $\bigcup$**Attach**$(Q, \mathbb{I} - (\mathcal{D} \setminus \{Q\}))$.

$(T_Q^i \mid 1 \le i \le k)$ is an enumeration of $\mathcal{D} \setminus \{Q\}$.

$X_Q^i = Q \cap T_Q^i$ (for $1 \le i \le k$).

A. $\mathcal{D}$ is an MNS set of $\mathbb{I}$ ***if and only if***, for all $Q \in \mathcal{D}$:

(1) A_Q is not contractible, but

(2) $A_Q \cup X_Q^{i_1} \cup \ldots \cup X_Q^{i_r}$ is contractible

for all <u>nonempty</u> subsets $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, k\}$.

Lemma 1: *For any finite collection $\mathcal{S}$ of polyhedra in* 3*-space, the following are equivalent*:

(a) $\bigcap \mathcal{T}$ *is contractible whenever* $\varnothing \ne \mathcal{T} \subseteq \mathcal{S}$.

(b) $\bigcup \mathcal{T}$ *is contractible whenever* $\varnothing \ne \mathcal{T} \subseteq \mathcal{S}$.

We now prove a series of 5 Claims which, together, constitute the "only if" parts of the main results.

Claim 1: $\bigcap \mathcal{D} = \varnothing \Rightarrow \mathcal{D}$ *cannot be MNS*

<u>**Proof**</u>: Suppose $\bigcap \mathcal{D} = \varnothing$ and $\mathcal{D}$ is MNS in an image $\mathbb{I}$. Pick any $Q \in \mathcal{D}$. Then, with the above notation,

$$X_Q^1 \cap \ldots \cap X_Q^k = Q \cap T_Q^1 \cap \ldots \cap T_Q^k = \bigcap \mathcal{D} = \varnothing$$

Let $\mathcal{S} = \{A_Q \cup X_Q^1, \ \ldots, \ A_Q \cup X_Q^k\}$.

Then $\bigcap \mathcal{S} = A_Q \cup (X_Q^1 \cap \ldots \cap X_Q^k) = A_Q$.

By (2): $\bigcup \mathcal{T}$ is contractible whenever $\varnothing \ne \mathcal{T} \subseteq \mathcal{S}$.

By (1): $\bigcap \mathcal{S} = A_Q$ is not contractible.

This contradiction of Lemma 1 proves Claim 1. //

Recall: For each $Q \in \mathcal{D}$: A_Q denotes the set $\bigcup$**Attach**$(Q, \mathbb{I} - (\mathcal{D} \setminus \{Q\}))$.

$(T_Q^i \mid 1 \le i \le k)$ is an enumeration of $\mathcal{D} \setminus \{Q\}$.

$X_Q^i = Q \cap T_Q^i$ (for $1 \le i \le k$).

A. $\mathcal{D}$ is an MNS set of $\mathbb{I}$ ***if and only if***, for all $Q \in \mathcal{D}$:

(1) A_Q is not contractible, but

(2) $A_Q \cup X_Q^{i_1} \cup \ldots \cup X_Q^{i_r}$ is contractible

for all <u>nonempty</u> subsets $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, k\}$.

Lemma 1: *For any finite collection $\mathcal{S}$ of polyhedra in 3-space, the following are equivalent*:

(a) $\bigcap \mathcal{T}$ *is contractible whenever* $\varnothing \ne \mathcal{T} \subseteq \mathcal{S}$.

(b) $\bigcup \mathcal{T}$ *is contractible whenever* $\varnothing \ne \mathcal{T} \subseteq \mathcal{S}$.

Claim 2: ($\mathcal{D}$ *is an MNS set of* $\mathbb{I} \wedge \bigcap \mathcal{D}$ *is a* 0-*xel*)

$\Rightarrow$ $\mathcal{D}$ *is a weak component of the* 1*'s of* $\mathbb{I}$.

<u>Proof</u>: Suppose $\bigcap \mathcal{D}$ is a 0-xel $\{v\}$, and $\mathcal{D}$ is an MNS set of an image $\mathbb{I}$ but $\mathcal{D}$ is ***not*** a weak component of $\mathbb{I}^{-1}[\{1\}]$. Pick $Q \in \mathcal{D}$ such that $A_Q \ne \varnothing$. Then:

$$X_Q^1 \cap \ldots \cap X_Q^k = Q \cap T_Q^1 \cap \ldots \cap T_Q^k = \bigcap \mathcal{D} = \{v\}$$

Let $\mathcal{S} = \{A_Q \cup X_Q^1, \ldots, A_Q \cup X_Q^k\}$, so $\bigcap \mathcal{S} = A_Q \cup (X_Q^1 \cap \ldots \cap X_Q^k)$.

Then $\bigcap \mathcal{S} = A_Q \cup \{v\}$; and <u>this set is ***not*** contractible</u> since:

- If $v \notin A_Q$, then $A_Q \cup \{v\}$ is disconnected, as $A_Q \ne \varnothing$.
- If $v \in A_Q$, then $A_Q \cup \{v\} = A_Q$ is not contractible, by (1).

But, by (2), $\bigcup \mathcal{T}$ is contractible whenever $\varnothing \ne \mathcal{T} \subseteq \mathcal{S}$.

This contradiction of Lemma 1 proves Claim 2. //

Recall:

> **Lemma 1:** *For any finite collection $\mathcal{S}$ of polyhedra in* 3*-space, the following are equivalent*:
>
> (a) $\bigcap\mathcal{T}$ *is contractible whenever* $\emptyset \neq \mathcal{T} \subseteq \mathcal{S}$.
>
> (b) $\bigcup\mathcal{T}$ *is contractible whenever* $\emptyset \neq \mathcal{T} \subseteq \mathcal{S}$.

From this lemma we can derive the following similar results, which deal with collections $\mathcal{S}$ all of whose nonempty ***proper*** subcollections have contractible unions or contractible intersections:

Cor. 1: *For any finite collection $\mathcal{S}$ of polyhedra in* 3*-space, the following are equivalent*:

(a) $\bigcup\mathcal{T}$ *is contractible whenever* $\emptyset \neq \mathcal{T} \subsetneq \mathcal{S}$.

(b) $\bigcap\mathcal{T}$ *is contractible whenever* $\emptyset \neq \mathcal{T} \subsetneq \mathcal{S}$.

Cor. 2: *For any finite collection $\mathcal{S}$ of polyhedra in* 3*-space, the following are equivalent*:

(a) $\bigcup\mathcal{T}$ *is contractible whenever* $\emptyset \neq \mathcal{T} \subsetneq \mathcal{S}$, *and* $\bigcap\mathcal{S}$ *is contractible.*

(b) $\bigcap\mathcal{T}$ *is contractible whenever* $\emptyset \neq \mathcal{T} \subsetneq \mathcal{S}$, *and* $\bigcup\mathcal{S}$ *is contractible.*

Cor. 1 is obtained by applying Lemma 1 to each nonempty proper subcollection $\mathcal{T}$ of $\mathcal{S}$. Cor. 2 is a straightforward consequence of Lemma 1 and Cor. 1.

Recall: For each $Q \in \mathcal{D}$: C_Q denotes the set $\bigcup$**Coattach**$(Q, \mathbb{I})$.

$(T_Q^i \mid 1 \le i \le k)$ is an enumeration of $\mathcal{D} \setminus \{Q\}$.

$X_Q^i = Q \cap T_Q^i$ (for $1 \le i \le k$).

B. $\mathcal{D}$ is an MNCS set of $\mathbb{I}$ ***if and only if***, for all $Q \in \mathcal{D}$:

(1') $C_Q \cup X_Q^1 \cup \ldots \cup X_Q^k$ is not contractible, but

(2') $C_Q \cup X_Q^{i_1} \cup \ldots \cup X_Q^{i_r}$ is contractible for all <u>*proper*</u> subsets $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, k\}$.

Cor. 2: *For any finite collection $\mathcal{S}$ of polyhedra in 3-space, the following are equivalent*:

(a) $\bigcup\mathcal{T}$ *is contractible whenever* $\emptyset \ne \mathcal{T} \subsetneq \mathcal{S}$, *and* $\bigcap\mathcal{S}$ *is contractible.*

(b) $\bigcap\mathcal{T}$ *is contractible whenever* $\emptyset \ne \mathcal{T} \subsetneq \mathcal{S}$, *and* $\bigcup\mathcal{S}$ *is contractible.*

Claim 3: $\bigcap\mathcal{D} = \emptyset \Rightarrow \mathcal{D}$ *cannot be MNCS*

<u>**Proof**</u>: Suppose $\bigcap\mathcal{D} = \emptyset$ and $\mathcal{D}$ is MNCS in an image $\mathbb{I}$. Pick any $Q \in \mathcal{D}$. Then, with the above notation,

$$X_Q^1 \cap \ldots \cap X_Q^k = Q \cap T_Q^1 \cap \ldots \cap T_Q^k = \bigcap\mathcal{D} = \emptyset$$

Let $\mathcal{S} = \{C_Q \cup X_Q^1, \ldots, C_Q \cup X_Q^k\}$, so $\bigcap\mathcal{S} = C_Q \cup (X_Q^1 \cap \ldots \cap X_Q^k) = C_Q$.

By (2'): $\bigcup\mathcal{T}$ is contractible whenever $\emptyset \ne \mathcal{T} \subsetneq \mathcal{S}$, and $\bigcap\mathcal{S} = C_Q$ is contractible.

By (1'): $\bigcup\mathcal{S} = C_Q \cup X_Q^1 \cup \ldots \cup X_Q^k$ is ***not*** contractible.

This contradiction of Cor. 2 proves Claim 3. //

Recall: For each $Q \in \mathcal{D}$: C_Q denotes the set $\bigcup$**Coattach**$(Q, \mathbb{I})$.

$(T_Q^i \mid 1 \le i \le k)$ is an enumeration of $\mathcal{D} \setminus \{Q\}$.

$X_Q^i = Q \cap T_Q^i$ (for $1 \le i \le k$).

B. $\mathcal{D}$ is an MNCS set of $\mathbb{I}$ ***if and only if***, for all $Q \in \mathcal{D}$:

(1') $C_Q \cup X_Q^1 \cup \ldots \cup X_Q^k$ is not contractible, but

(2') $C_Q \cup X_Q^{i_1} \cup \ldots \cup X_Q^{i_r}$ is contractible

for all <u>*proper*</u> subsets $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, k\}$.

Cor. 1: *For any finite collection $\mathcal{S}$ of polyhedra in* 3*-space, the following are equivalent*:

(a) $\bigcup\mathcal{T}$ *is contractible whenever* $\emptyset \ne \mathcal{T} \subsetneq \mathcal{S}$.

(b) $\bigcap\mathcal{T}$ *is contractible whenever* $\emptyset \ne \mathcal{T} \subsetneq \mathcal{S}$.

Claim 4: $\exists \mathcal{D}' \; (\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap\mathcal{D}' = \bigcap\mathcal{D})$

$\Rightarrow \mathcal{D}$ *cannot be MNCS*

<u>**Proof**</u> **(part 1)**: Suppose $\mathcal{D}'$ satisfies $\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap\mathcal{D}' = \bigcap\mathcal{D}$ and $\mathcal{D}$ is MNCS in an image $\mathbb{I}$. Pick any $Q \in \mathcal{D}'$.

WLOG $\mathcal{D}' = \{Q, T_Q^2, \ldots, T_Q^k\}$. Then, since $\bigcap\mathcal{D}' = \bigcap\mathcal{D}$,

$$X_Q^1 \cap \ldots \cap X_Q^k = Q \cap T_Q^1 \cap \ldots \cap T_Q^k = Q \cap T_Q^2 \cap \ldots \cap T_Q^k = X_Q^2 \cap \ldots \cap X_Q^k$$

Let $\mathcal{S} = \{C_Q \cup X_Q^1, \ldots, C_Q \cup X_Q^k\}$, $\mathcal{S}' = \{C_Q \cup X_Q^2, \ldots, C_Q \cup X_Q^k\}$.

Now $\mathcal{S}' \subsetneq \mathcal{S}$; by (2', 1'), $\bigcup\mathcal{S}'$ is contractible but $\bigcup\mathcal{S}$ is not.

By (2'): $\bigcup\mathcal{T}$ is contractible whenever $\emptyset \ne \mathcal{T} \subsetneq \mathcal{S}$.
So, by Cor. 1, $\bigcap\mathcal{T}$ is contractible whenever $\emptyset \ne \mathcal{T} \subsetneq \mathcal{S}$. (*)

Recall: B. $\mathcal{D}$ is an MNCS set of $\mathbb{I}$ ***if and only if***, for all $Q \in \mathcal{D}$:

(1') $C_Q \cup X_Q^1 \cup \ldots \cup X_Q^k$ is not contractible, but

(2') $C_Q \cup X_Q^{i_1} \cup \ldots \cup X_Q^{i_r}$ is contractible

for all <u>proper</u> subsets $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, k\}$.

Claim 4: $\exists \mathcal{D}'\ (\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap \mathcal{D}' = \bigcap \mathcal{D}) \Rightarrow \mathcal{D}$ *cannot be MNCS*

Proof (part 1): Suppose $\exists \mathcal{D}'\ (\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap \mathcal{D}' = \bigcap \mathcal{D})$ and $\mathcal{D}$ is MNCS in an image $\mathbb{I}$. Pick any $Q \in \mathcal{D}'$. WLOG $\mathcal{D}' = \{Q, T_Q^2, \ldots, T_Q^k\}$. Then

$$X_Q^1 \cap \ldots \cap X_Q^k = Q \cap T_Q^1 \cap \ldots \cap T_Q^k = Q \cap T_Q^2 \cap \ldots \cap T_Q^k = X_Q^2 \cap \ldots \cap X_Q^k.$$

Let $\mathcal{S} = \{C_Q \cup X_Q^1, \ldots, C_Q \cup X_Q^k\}$, $\mathcal{S}' = \{C_Q \cup X_Q^2, \ldots, C_Q \cup X_Q^k\}$.

Now $\mathcal{S}' \subsetneq \mathcal{S}$; by (2', 1'), $\bigcup \mathcal{S}'$ is contractible but $\bigcup \mathcal{S}$ is not.

By (2'): $\bigcup \mathcal{T}$ is contractible whenever $\varnothing \neq \mathcal{T} \subsetneq \mathcal{S}$.

So, by Cor. 1, $\bigcap \mathcal{T}$ is contractible whenever $\varnothing \neq \mathcal{T} \subsetneq \mathcal{S}$. (*)

<u>**Proof of Claim 4**</u> **(part 2)**: We have just shown

$\bigcap \mathcal{T}$ is contractible whenever $\varnothing \neq \mathcal{T} \subsetneq \mathcal{S}$ (*)

In particular, $\bigcap \mathcal{S}'$ is contractible.

But $\bigcap \mathcal{S} = C_Q \cup (X_Q^1 \cap \ldots \cap X_Q^k) = C_Q \cup (X_Q^2 \cap \ldots \cap X_Q^k) = \bigcap \mathcal{S}'$.

Hence $\bigcap \mathcal{S}$ is contractible, and so, by (*),

$\bigcap \mathcal{T}$ is contractible whenever $\varnothing \neq \mathcal{T} \subseteq \mathcal{S}$.

Lemma 1 now tells us

$\bigcup \mathcal{T}$ is contractible whenever $\varnothing \neq \mathcal{T} \subseteq \mathcal{S}$.

In particular, $\bigcup \mathcal{S}$ is contractible.

This contradiction of (1') proves Claim 4. //

Recall: For each $Q \in \mathcal{D}$: C_Q denotes the set $\bigcup$**Coattach**$(Q, \mathbb{I})$.

$(T_Q^i \mid 1 \le i \le k)$ is an enumeration of $\mathcal{D} \setminus \{Q\}$.

$X_Q^i = Q \cap T_Q^i$ (for $1 \le i \le k$).

B. $\mathcal{D}$ is an MNCS set of $\mathbb{I}$ ***if and only if***, for all $Q \in \mathcal{D}$:

(1') $C_Q \cup X_Q^1 \cup \ldots \cup X_Q^k$ is not contractible, but

(2') $C_Q \cup X_Q^{i_1} \cup \ldots \cup X_Q^{i_r}$ is contractible

for all <u>*proper*</u> subsets $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, k\}$.

Claim 5: *If* $\mathcal{D}$ *is an MNCS set of an image* $\mathbb{I}$ *on an nD xel complex* ($n \le 4$)*, and* $|\mathcal{D}| = n+1$*, then* $\mathcal{D}$ *is a strong component of the* 1*'s of* $\mathbb{I}$.

<u>**Proof**</u>: Suppose $\mathcal{D}$ is an MNCS set of 1's in some image $\mathbb{I}$ on an nD xel complex, and $|\mathcal{D}| = n+1$.

Then, by Claim 3, $\bigcap \mathcal{D} \neq \emptyset$

and, by Claim 4, $\nexists \mathcal{D}' \; (\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap \mathcal{D}' = \bigcap \mathcal{D})$.

Pick any $Q \in \mathcal{D}$, and define T_Q^i and X_Q^i $(1 \le i \le k = n)$ as before.

As we noted earlier, the condition $\nexists \mathcal{D}' \; (\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap \mathcal{D}' = \bigcap \mathcal{D})$ implies that no two of the n+1 members of the chain

$$Q \supseteq Q \cap T_Q^1 \supseteq Q \cap T_Q^1 \cap T_Q^2 \supseteq \ldots \supseteq Q \cap T_Q^1 \cap \ldots \cap T_Q^n = \bigcap \mathcal{D}$$

are equal, so each member of this chain after the first has strictly lower dimensionality than its predecessor.

Hence $\bigcap \mathcal{D}$ is a 0-xel (since $\dim(Q) = n$ and $\bigcap \mathcal{D} \neq \emptyset$).

<u>Proof of Claim 5</u> (continued): Let $\mathcal{S} = \{C_Q, X_Q^1, \ldots, X_Q^k\}$, and let $\mathcal{S}' = \{X_Q^1, \ldots, X_Q^k\}$. Then $\bigcap \mathcal{S}' = \bigcap \mathcal{D} \neq \varnothing$, so that $\bigcap \mathcal{T}$ is a xel (and is contractible) whenever $\varnothing \neq \mathcal{T} \subseteq \mathcal{S}'$.
Hence, by Lemma 1,

$$\bigcup \mathcal{T} \text{ is contractible whenever } \varnothing \neq \mathcal{T} \subseteq \mathcal{S}'. \qquad (\star)$$

By (1', ⋆, 2'), $\bigcup \mathcal{S}$ is not contractible, but

$$\bigcup \mathcal{T} \text{ is contractible whenever } \varnothing \neq \mathcal{T} \subsetneq \mathcal{S}. \qquad (*)$$

So, by Cor. 2, $\bigcap \mathcal{S} = C_Q \cap \bigcap \mathcal{D}$ is *not* contractible, whence

$$\bigcap \mathcal{S} = \varnothing \qquad (\blacklozenge)$$

as the 0-xel $\bigcap \mathcal{D}$ has no nonempty non-contractible subset.
Moreover, by (*) and Cor. 1,

$$\bigcap \mathcal{T} \text{ is contractible whenever } \varnothing \neq \mathcal{T} \subsetneq \mathcal{S} \qquad (\clubsuit)$$

As $|\mathcal{S}| = n+1$, it follows from (♦), (♣), and a result of topology known as the Nerve Theorem that $\bigcup \mathcal{S}$ is *homotopy equivalent to an* (n–1)-*dimensional sphere*.

However, $\bigcup \mathcal{S}$ is the union of a subset of **bdryfaces**(Q), and the only such union that is homotopy equivalent to an (n–1)-dimensional sphere is $\bigcup$**bdryfaces**(Q) itself.

Hence $\bigcup \mathcal{S} = \bigcup$**bdryfaces**($Q$) and, since $\bigcup \mathcal{S} = \bigcup$**Coattach**($Q$, $\mathbb{I} - (\mathcal{D} \setminus \{Q\})$), we see that Q is *not* strongly adjacent to any 1 of $\mathbb{I} - (\mathcal{D} \setminus \{Q\})$.

As Q is an arbitrary element of $\mathcal{D}$, it follows that $\mathcal{D}$ is a strong component of the 1's of $\mathbb{I}$. //

Concluding Remarks 1

The concepts of ***minimal non-simple*** (***MNS***) and ***minimal non-cosimple*** (***MNCS***) set provide the basis for a powerful method of establishing that a proposed parallel thinning algorithm "preserves topology".

For binary images on the grid-cells of a complex **K**, the method depends on knowing the answers to the following questions:

For algorithms that are expected to preserve weak components of 1's and strong components of 0's:

- Which sets of grid-cells can be MNS on **K**?
- Which sets of grid-cells can be MNS on **K** as a proper subset of a weak component of the 1's?

For algorithms that are expected to preserve strong components of 1's and weak components of 0's:

- Which sets of grid-cells can be MNCS on **K**?
- Which sets of grid-cells can be MNCS on **K** as a proper subset of a strong component of the 1's?

Concluding Remarks 2

The above questions have been answered in the literature for the following 5 xel complexes:

- 2D, 3D, and 4D cubical xel complexes
- 2D hexagonal xel complex
- 3D face-centered cubical xel complex

Our main results *unify* and *generalize* this earlier work, by answering the questions for all xel complexes of dimension ≤ 4, as follows:

Theorem *Let* **K** *be an nD xel complex* (*where* $n \leq 4$) *and let* $\mathcal{D}$ *be a nonempty set of grid-cells of* **K**. *Then*:

A1. $\mathcal{D}$ *can be MNS on* **K** ***iff*** $\bigcap \mathcal{D} \neq \emptyset$.

A2. $\mathcal{D}$ *can be MNS on* **K** *as a* <u>*proper*</u> *subset of a weak component* ***iff*** $\dim(\bigcap \mathcal{D}) \geq 1$.

B1. $\mathcal{D}$ *can be MNCS on* **K**
iff $\bigcap \mathcal{D} \neq \emptyset$ *and* $\nexists \mathcal{D}' \ (\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap \mathcal{D}' = \bigcap \mathcal{D})$.
[**<u>Note</u>**: This condition implies $|\mathcal{D}| \leq n+1$.]

B2. $\mathcal{D}$ *can be MNCS on* **K** *as a* <u>*proper*</u> *subset of a strong component*
iff $|\mathcal{D}| \leq n$, $\bigcap \mathcal{D} \neq \emptyset$ *and* $\nexists \mathcal{D}' \ (\mathcal{D}' \subsetneq \mathcal{D} \wedge \bigcap \mathcal{D}' = \bigcap \mathcal{D})$.